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A CONSTRUCTION OF LAGRANGIAN SUBMANIFOLDS IN COMPLEX EUCLIDEAN SPACES WITH LEGENDRE CURVES

YUN MYUNG OH

ABSTRACT. In [1], B. Y. Chen provided a new method to construct Lagrangian surfaces in \mathbf{C}^2 by using Legendre curves in $S^3(1) \subset \mathbf{C}^2$. In this paper, we investigate the similar methods to construct some Lagrangian submanifolds in complex Euclidean spaces $\mathbf{C}^n (n \geq 3)$.

1. INTRODUCTION

A regular curve $z : I \rightarrow S^{2n-1}(r) \subset \mathbf{C}^n$ in the hypersphere $S^{2n-1}(r)$ of radius r centered at the origin of \mathbf{C}^n is called a Legendre curve if $\langle z'(t), iz(t) \rangle = 0$ identically. The idea of special Legendre curves was introduced by B. Y. Chen in his paper [2]. The similar notion was introduced by the author, in [6], to find the explicit construction of Lagrangian isometric immersion of a real-space-form $M^n(c)$ into a complex-space-form $\widetilde{M}^n(4c)$. For $l = 1, \dots, k$, let $z : I_1 \times \dots \times I_k \rightarrow S^{2n-1}(1) \hookrightarrow \mathbf{C}^n$ be a sum of l unit speed Legendre curves $z_i : I_i \rightarrow S^{2n-1}(r_i) \in \mathbf{C}^n$ and a \mathbf{C}^n -valued function z_{l+1} of the variables t_{l+1}, \dots, t_k , i.e.

$$z = z_1(t_1) + z_2(t_2) + \dots + z_l(t_l) + z_{l+1}(t_{l+1}, t_{l+2}, \dots, t_k)$$

such that $\{z'_1, \dots, z'_l, \frac{\partial z_{l+1}}{\partial t_{l+1}}, \dots, \frac{\partial z_{l+1}}{\partial t_k}\}$ spans tangent space satisfying

$$\langle z, iz'_1 \rangle = \langle z, iz'_2 \rangle = \dots = \langle z, iz'_l \rangle = 0.$$

We define that z is an l -th Legendre translation submanifold in $S^{2n-1}(1) \subset \mathbf{C}^n$. Since $\{z'_j(t_j)\}_{j=1}^l$ are orthonormal tangent vector fields, we can choose $k-l$ orthonormal vector fields $\tilde{z}_{l+1}, \dots, \tilde{z}_k$ by taking Gram-Schmidt process to the tangent vector fields $\{\frac{\partial z_{l+1}}{\partial t_{l+1}}, \dots, \frac{\partial z_{l+1}}{\partial t_k}\}$ and thus $\{z'_1, \dots, z'_l, \tilde{z}_{l+1}, \dots, \tilde{z}_k\}$ are orthonormal tangent frame fields.

Hence, $z, iz, z'_1, \dots, z'_l, \tilde{z}_{l+1}, \dots, \tilde{z}_k, iz'_1, \dots, iz'_l, i\tilde{z}_{l+1}, \dots, i\tilde{z}_k$ form orthonormal vector fields defined on $I_1 \times \dots \times I_k$. Thus there exist orthonormal normal vector fields A_{k+2}, \dots, A_n defined along z such that $z, iz, z'_1, \dots, z'_l, \tilde{z}_{l+1}, \dots, \tilde{z}_k, iz'_1, \dots, iz'_l, i\tilde{z}_{l+1}, \dots, i\tilde{z}_k, A_{k+2}, \dots, A_n, iA_{k+2}, \dots, iA_n$ form an orthonormal basis on \mathbf{C}^n . With respect to the above orthonormal frame fields,

$$z''_j = -z_j + i\lambda_j z'_j - \sum_{\alpha} a_{\alpha} A_{\alpha} + \sum_{\beta} b_{\beta} iA_{\beta},$$

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where $a_\alpha = a_\alpha(t_1, \dots, t_k)$, $b_\beta = b_\beta(t_1, \dots, t_k)$ are real valued functions. If this expression can be reduced to

$$(1.1) \quad z_j'' = -z_j + i\lambda_j z_j' - \sum_{\alpha} a_{j,\alpha} A_{j,\alpha},$$

where $A_\alpha = A_{1,\alpha}(t_1) + \dots + A_{k,\alpha}(t_k)$ are associated orthonormal normal vector fields and $A'_{j,\alpha} = a_{j,\alpha}(t_j) z_j'(t_j)$ for some real valued functions $a_{j,\alpha} = a_{j,\alpha}(t_j)$ ($j = 1, \dots, l$), then $z = z(t_1, \dots, t_k)$ is called an l th special Legendre translation submanifold in $S^{2n-1}(1) \hookrightarrow \mathbf{C}^n$. The curvature of the curves z_j are defined as $\lambda_j = \langle z_j'', iz_j' \rangle$. Here, we can easily notice that if $k = 1$, then $z(t)$ becomes a special Legendre curve as in [2]. Also, we note here that every $(n-1)$ -th Legendre translation submanifold in $S^{2n-1} \subset \mathbf{C}^n$ is special.

We need the following lemma.

Lemma 1. [1] *Let A, B be two vectors in \mathbf{C}^n and z, w be two complex numbers. Then we have*

$$(1.2) \quad \begin{aligned} \langle zA, wB \rangle &= \langle z, w \rangle \langle A, B \rangle + \langle iz, w \rangle \langle A, iB \rangle, \\ \langle zA, iwB \rangle &= \langle z, w \rangle \langle A, iB \rangle + \langle z, iw \rangle \langle A, B \rangle, \end{aligned}$$

where $\langle z, w \rangle = \text{Real}(z\bar{w})$ denote the real part of the complex number $z\bar{w}$, \bar{w} the complex conjugate of w , and $\langle A, B \rangle$ denotes the canonical inner product of the vector A and B in the complex Euclidean n -plane \mathbf{C}^n .

As a converse of the definition of Legendre translation submanifold, we have the following proposition.

Proposition 1. *If $z = z_1(t_1) + \dots + z_{n-1}(t_{n-1}) : I_1 \times \dots \times I_{n-1} \rightarrow S^{2n-1}(1) \subset \mathbf{C}^n$ is a sum of unit speed Legendre curves satisfying*

$$(1.3) \quad z_j'' = i\lambda_j z_j' - z$$

for some nonzero real valued functions $\lambda_j, j = 1, \dots, n-1$, then z is an $(n-1)$ -th Legendre translation submanifold.

Proof. It suffices to show that $\langle z, iz_j' \rangle = 0, j = 1, \dots, n-1$. Since $\langle z, z_j' \rangle = 0$, and $\langle z, z_j'' \rangle = -1$, we have $\langle z, i\lambda_j z_j' - z \rangle = -1$ and then $\lambda_j \langle z, iz_j' \rangle = 0$. Since λ_j is nonzero, $\langle z, iz_j' \rangle = 0$ identically for all $j = 1, \dots, n-1$. \square

Next, we provide the example of a k -th special Legendre translation submanifold in $S^{2n+1} \subset \mathbf{C}^{n+1}$. Let $\lambda_1, \dots, \lambda_k (k \geq 2)$, $a_{1,k+2}, \dots, a_{1,n+1}, \dots, a_{k,k+2}, \dots, a_{k,n+1}$ be real numbers such that

$$\begin{aligned} \lambda_j &> 0, \quad 1 + \sum_{\alpha=k+2}^{n+1} a_{i,\alpha} a_{j,\alpha} = 0, \\ 1 &= \frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_k} \quad \text{for } \gamma_j = 1 + \sum_{\alpha=k+2}^{n+1} a_{j,\alpha}^2, \\ \mu_j^2 &= \lambda_j^2 + 4\gamma_j \quad \text{for } 1 \leq j \neq i \leq k. \end{aligned}$$

Then

$$\begin{aligned}
z &= z_1(t_1) + \dots + z_k(t_k) \\
&= \frac{\mu_1 - \lambda_1}{2\mu_1\gamma_1} \left(\frac{2\gamma_1}{\mu_1 - \lambda_1}, 0, \dots, 1, a_{1,k+2}, \dots, a_{1,n+1} \right) e^{\frac{\lambda_1 + \mu_1}{2} it_1} \\
&\quad + \frac{\mu_1 + \lambda_1}{2\mu_1\gamma_1} \left(\frac{-2\gamma_1}{\mu_1 + \lambda_1}, 0, \dots, 1, a_{1,k+2}, \dots, a_{1,n+1} \right) e^{\frac{\lambda_1 - \mu_1}{2} it_1} \\
&\quad + \dots + \frac{\mu_j - \lambda_j}{2\mu_j\gamma_j} \left(0, \dots, \frac{2\gamma_j}{\mu_j - \lambda_j}, 0, \dots, 1, a_{j,k+2}, \dots, a_{j,n+1} \right) e^{\frac{\lambda_j + \mu_j}{2} it_j} \\
&\quad + \frac{\mu_j + \lambda_j}{2\mu_j\gamma_j} \left(0, \dots, \frac{-2\gamma_j}{\mu_j + \lambda_j}, 0, \dots, 1, a_{j,k+2}, \dots, a_{j,n+1} \right) e^{\frac{\lambda_j - \mu_j}{2} it_j} \\
&\quad + \dots + \frac{\mu_k - \lambda_k}{2\mu_k\gamma_k} \left(0, \dots, \frac{2\gamma_k}{\mu_k - \lambda_k}, 1, a_{k,k+2}, \dots, a_{k,n+1} \right) e^{\frac{\lambda_k + \mu_k}{2} it_k} \\
&\quad + \frac{\mu_k + \lambda_k}{2\mu_k\gamma_k} \left(0, \dots, \frac{-2\gamma_k}{\mu_k + \lambda_k}, 1, a_{k,k+2}, \dots, a_{k,n+1} \right) e^{\frac{\lambda_k - \mu_k}{2} it_k}
\end{aligned}$$

defines a special Legendre translation submanifold in $S^{2n+1}(1) \hookrightarrow \mathbf{C}^{n+1}$, that is, it satisfies

$$\begin{aligned}
z_j'' &= -z_j + i\lambda_j z_j' - \sum a_{j,\alpha} A_{j,\alpha} \\
\langle z, iz_j' \rangle &= 0 \text{ for } j = 1, \dots, k,
\end{aligned}$$

where $A_\alpha = A_{1,\alpha} + \dots + A_{k,\alpha}$ are some associated orthonormal normal vector fields satisfying $A_{j,\alpha} = a_{j,\alpha} z_j$ for $j = 1, \dots, k$ and $\alpha = k + 2, \dots, n + 1$.

2. MAIN RESULTS

In this chapter, we construct some Lagrangian submanifolds in complex Euclidean space \mathbf{C}^n using an $(n - 1)$ special Legendre translation submanifold.

Theorem 1. *Let $f : I_1 \rightarrow \mathbf{C}^*$ be a regular curve defined on an open interval I_1 and $z : I_2 \times \dots \times I_n \rightarrow S^{2n-1}(r) \subset \mathbf{C}^n$ be a sum of unit speed Legendre curves z_j in $S^{2n-1}(r_j)$ defined on an open interval I_j . Then we have the following.*

(a) *If $z = z_2(t_2) + z_3(t_3) + \dots + z_n(t_n)$ is an $(n - 1)$ Legendre translation submanifold, then, for any function $p_j : I_j \rightarrow \mathbf{C}, j = 2, \dots, n$ the map*

$$(2.1) \quad L(s, t_2, \dots, t_n) = f(s)z(t_2, \dots, t_n) - \sum_{j=2}^n \int_0^{t_j} p_j(t) z_j'(t) dt$$

is a Lagrangian isometric immersion of $M^n = (U, g)$ into \mathbf{C}^n , where the set U is defined as

$$U := \{(s, t_2, \dots, t_n) \in I_1 \times I_2 \times \dots \times I_n : f(s) \neq p_j(t_j) \text{ for all } j = 2, \dots, n\}$$

and the metric g is the induced metric given by

$$(2.2) \quad g = r^2 |f'(s)|^2 ds^2 + \sum_{j=2}^n |f(s) - p_j(t_j)|^2 dt_j^2$$

(b) *Conversely, if f does not contain any circular arcs, and if L as in (2.1) is a Lagrangian immersion, then $z = z_2(t_2) + \dots + z_n(t_n) : I_2 \times \dots \times I_n \rightarrow S^{2n-1}(r) \subset \mathbf{C}^n$ is an $(n - 1)$ Legendre translation submanifold.*

Proof. Let $f : I_1 \rightarrow \mathbf{C}^*$ be a regular curve defined on an open interval I_1 and $z : I_2 \times \dots \times I_n \rightarrow S^{2n-1}(r) \subset \mathbf{C}^n$ be a smooth \mathbf{C}^n -valued map defined on a product of open intervals I_2, \dots, I_n . Using (2.1), we have

$$(2.3) \quad L_s = \frac{\partial L}{\partial s} = f'(s)z(t_2, \dots, t_n), \quad L_{t_j} = (f(s) - p_j(t_j))z_j'(t_j), \quad j = 2, \dots, n.$$

Now, applying Lemma 1 and using (2.3) yield

$$(2.4) \quad \begin{cases} \langle L_s, L_s \rangle = r^2 |f'(s)|^2, \\ \langle L_s, L_{t_j} \rangle = \langle if', f(s) - p_j(t_j) \rangle \langle z, iz'_j(t_j) \rangle, \\ \langle L_{t_j}, L_{t_j} \rangle = |f(s) - p_j(t_j)|^2, j = 2, \dots, n \\ \langle L_{t_j}, L_{t_k} \rangle = \langle i(f(s) - p_j), f(s) - p_k(t_k) \rangle \langle z_j, iz'_k \rangle, j \neq k = 2, \dots, n \end{cases}$$

Since z is an $(n-1)$ Legendre translation submanifold, we can find the induced metric g on U given as in (2.2) from (2.4) and also, $\langle L_s, iL_{t_j} \rangle = \langle L_{t_j}, iL_{t_k} \rangle = 0$ for all $j, k = 2, \dots, n$ which imply that the map L is Lagrangian.

For (b), suppose L , defined in (2.2) is a Lagrangian isometric immersion. The similar computation shows that $\langle L_s, iL_{t_j} \rangle = \langle f'(s), f(s) - p_j(t_j) \rangle \langle z, iz'_j(t_j) \rangle \equiv 0$ identically. If there exist one j such that $\langle f'(s), f(s) - p_j(t_j) \rangle = 0$ for all s in an open subinterval $I_0 \subset I_1$, then $\frac{d}{ds}|f(s) - p_j(t_j)|^2 = 0$ which means for each $t_j \in I_j$, the curve f is contained in a circle centered at $p_j(t_j)$. It is impossible so that if f does not contain any circular arcs, then we have $\langle z, iz'_j(t_j) \rangle \equiv 0$ for all $j = 2, \dots, n$ and thus z becomes an $(n-1)$ Legendre translation submanifold. \square

The next theorem shows the extrinsic properties of the immersion.

Theorem 2. *Let $f : I_1 \rightarrow \mathbf{C}^*$ be a unit speed curve, $z : I_2 \times \dots \times I_n \rightarrow S^{2n-1}(1) \subset \mathbf{C}^n$ an $(n-1)$ th Legendre translation submanifold, $p_j : I_j \rightarrow \mathbf{C}$, $j = 2, \dots, n$ complex valued functions, and $L : (U, g) \rightarrow \mathbf{C}^n$ be the Lagrangian isometric immersion defined by*

$$(2.5) \quad L(s, t_2, \dots, t_n) = f(s)z(t_2, \dots, t_n) - \sum_{j=2}^n \int_0^{t_j} p_j(t)z'_j(t)dt.$$

Then we find

- (a) L_s is an eigenvector of the shape operator A_{JL_s} with eigenvalue κ , where κ is the curvature function of f .
- (b) For $j = 2, \dots, n$, L_{t_j} is an eigenvector of the shape operator $A_{JL_{t_j}}$ if and only if $p_2 = \dots = p_n = p$ are constants and $f(s) = cs + p$ for some $c \in \mathbf{C}$ with $|c| = 1$.
- (c) L is totally geodesic if and only if $n = 2$, $p_2 = p$ is a constant, $f(s) = cs + p$, $|c| = 1$, z is a great circle in $S^3(1)$ and $L = (f - p)z = csz$ for a constant c .

Proof. From (2.5), we have

$$(2.6) \quad \begin{aligned} L_{ss} &= f''(s)z, L_{st_j} = f'(s)z'_j, \\ L_{t_j t_j} &= -p'_j z'_j + (f - p_j)z''_j, L_{t_i t_j} = 0, i \neq j = 2, \dots, n, \end{aligned}$$

By applying Lemma 1 to (2.6), we obtain

$$(2.7) \quad \begin{cases} \langle L_{ss}, iL_s \rangle = \langle f''(s)z, if'z \rangle = \kappa, \\ \langle L_{ss}, iL_{t_j} \rangle = \langle L_{st_j}, iL_s \rangle = \langle L_{st_j}, iL_{t_k} \rangle = 0, k \neq j \\ \langle L_{t_j t_j}, iL_s \rangle = \langle L_{st_j}, iL_{t_j} \rangle = \langle f', i(f - p_j) \rangle, \\ \langle L_{t_j t_j}, iL_{t_j} \rangle = k_j(t_j) \|f - p_j\|^2 - \langle p'_j, i(f - p_j) \rangle, j = 2, \dots, n, \end{cases}$$

where $k_j(t_j) = \langle z''_j, iz'_j \rangle$ is the curvature function of the curve z_j . Let $e_1 = L_s$, $e_j = \frac{L_{t_j}}{|f - p_j|}$, $j = 2, \dots, n$. Then e_1, e_2, \dots, e_n are orthonormal frame fields. Therefore, the

second fundamental form h is

$$(2.8) \quad \begin{aligned} h(e_1, e_1) &= \kappa(s)Je_1, \\ h(e_1, e_j) &= \mu_j Je_j, \\ h(e_j, e_j) &= \mu_j Je_1 + \alpha_j Je_j, \\ h(e_i, e_j) &= 0, \text{ for } i \neq j = 2, \dots, n, \end{aligned}$$

where $\mu_j = \frac{\langle f', i(f - p_j) \rangle}{|f - p_j|^2}$, $\alpha_j = \frac{1}{|f - p_j|^3}(\kappa_j(t_j)|f - p_j|^2 - \langle p'_j, i(f - p_j) \rangle)$ and $\kappa_j = \langle z'_j, iz'_j \rangle$ for $j = 2, \dots, n$. We can easily see that L_s is an eigenvector of A_{JL_s} with eigenvalue κ which is the curvature function of f .

For (b), L_{t_j} is an eigenvector of $A_{JL_{t_j}}$ if and only if $\langle f', i(f - p_j) \rangle = 0$ for $j = 2, \dots, n$

which implies that the position vector $\gamma_j(s) = f(s) - p_j(t_j)$ is always tangent to the curve γ_j for any fixed t_j . Thus, for each t_j , γ_j is a part of a line through 0 of \mathbf{C} . Therefore, there exist unit vector fields c_j in \mathbf{C} such that $\gamma_j(s) = f(s) - p_j(t_j) = c_j(t_j)s$ which yields $c_2(t_2) = \dots = c_n(t_n) = c$, $p_2 = \dots = p_n = p$, where c and p are constants in \mathbf{C} . Thus, $f(s) = cs + p$ and $|c| = 1$.

Suppose L is totally geodesic. From the second statement (b), we know that $\kappa = 0$.

By (2.8), it suffices to show that $\alpha_j = 0$ for $j = 2, \dots, n$ which is equivalent to $\kappa_j(t_j) = 0$. It is impossible unless $n = 2$ which means that $z : I_2 \rightarrow S^3(1) \subset \mathbf{C}^2$ is a great circle in $S^3(1)$. \square

3. APPLICATION

The following result shows examples of Lagrangian submanifolds in complex Euclidean space using the main results discussed before.

Theorem 3. *Let $f : I_1 \rightarrow \mathbf{C}^*$ be a unit speed curve, $z : I_2 \times \dots \times I_n \rightarrow S^{2n-1}(1) \subset \mathbf{C}^n$ a $(n-1)$ -th special Legendre translation submanifold, $p_j : I_j \rightarrow \mathbf{C}$, $j = 2, \dots, n$ complex valued functions. Then $L : (U, g) \rightarrow \mathbf{C}^n$, defined as in (2.5), is minimal if and only if, up to rigid motions of \mathbf{C}^n , one of the following holds:*

(a) *If $n = 2$, then L is either a totally geodesic immersion or an open portion of the Lagrangian catenoid, up to dilations and rigid motions.*

(b) *If $n \geq 3$, then $L = (f - p) \otimes z$ is a complex extensor where $f(s(x)) = p + x + iy(x)$ is a unit speed curve satisfying a differential equation*

$$(y - xy')^{n+1} = c(y'')^{n-1}(1 + y'(x)^2)^{2-n}$$

for a constant c and $z = z_2 + \dots + z_n$ is a sum of circles z_j 's in \mathbf{C}^n .

Proof. The induced metric on U is

$$g = ds^2 + \sum_2^n |f(s) - p_j(t_j)|^2 dt_j^2.$$

Then $e_1 = L_s$, $e_j = \frac{L_{t_j}}{|f - p_j|}$, $j = 2, \dots, n$ are orthonormal frame fields and using these frames and the induced metric g , we have

$$(3.1) \quad \begin{aligned} \nabla_{e_1} e_1 &= \nabla_{e_1} e_j = \nabla_{e_i} e_j = 0, \text{ for } i \neq j = 2, \dots, n, \\ \nabla_{e_j} e_j &= \frac{-\langle f', f - p_j \rangle}{|f - p_j|^2} e_1, \\ \nabla_{e_j} e_1 &= \frac{\langle f', f - p_j \rangle}{|f - p_j|^2} e_j, j = 2, \dots, n. \end{aligned}$$

Using the Codazzi equation and (2.8), we have, for $j = 2, \dots, n$, $(\nabla_{e_j} h)(e_1, e_j) = (\nabla_{e_1} h)(e_j, e_j)$ which gives

$$(3.2) \quad e_1(\mu_j) = (\kappa - 2\mu_j) \frac{\langle f', f - p_j \rangle}{|f - p_j|^2}$$

$$(3.3) \quad e_1(\alpha_j) = e_j(\mu_j) - \alpha_j \frac{\langle f', f - p_j \rangle}{|f - p_j|^2}$$

Because of the minimality condition, (2.8) implies

$$(3.4) \quad \kappa + \mu_2 + \dots + \mu_n = 0, e_j(\mu_j) = 0, \alpha_j = 0, j = 2, \dots, n$$

and then

$$(3.5) \quad \kappa_j(t_j)|f(s) - p_j|^2 - \langle p'_j, i(f - p_j) \rangle = 0.$$

$$(3.6) \quad (\kappa - 2\mu_j) \langle p'_j, f' \rangle = 0$$

By differentiating the equation (3.5) with respect to s , we obtain

$$(3.7) \quad \langle i f', p'_j \rangle = 2\kappa_j \langle f', f(s) - p_j \rangle$$

Another differentiating the equation (3.7) with respect to s and replacing f'' by $i\kappa f'$ yields

$$(3.8) \quad \kappa \langle f', p'_j \rangle + 2\kappa_j + 2\kappa\kappa_j \langle i f', f(s) - p_j \rangle = 0.$$

Now, we can consider two cases.

Case (a) Suppose $\kappa = 0$. It is immediate to know that $\kappa_j = 0$ for all $j = 2, \dots, n$ from (3.8). Since $\alpha_j = 0, j = 2, \dots, n$, (3.3) and (3.7) imply that

$$0 = \frac{\partial}{\partial t_j} \left(\frac{\langle f', i(f - p_j) \rangle}{|f - p_j|^2} \right) = \mu_j \frac{2 \langle f - p_j, p'_j \rangle}{|f - p_j|^2}$$

If there exist $\mu_j \neq 0$, then the above equation implies $\langle f - p_j, p'_j \rangle = 0$ and thus $|f(s) - p_j(t_j)|_{t_j}^2 = 0$ which means that for each s , every curve p_j is contained in a circle centered at $f(s)$ which is impossible. Therefore, we can conclude that for all $j = 2, \dots, n$, $\mu_j = 0$. Therefore, L is totally geodesic and then by theorem 2, n must be 2.

Case (b) Assume that $\kappa \neq 0$. Differentiating (3.8) with respect to s and replacing f'' by $i\kappa f'$, we find that

$$0 = \kappa^2 \langle i f', p'_j \rangle - 2\kappa^2 \kappa_j \langle f', f - p_j \rangle + \kappa' \langle f', p'_j \rangle + 2\kappa' \kappa_j \langle i f', f - p_j \rangle$$

By applying (3.7) and (3.8), the above becomes that

$$\begin{aligned} & 2\kappa^2 \kappa_j \langle f', f - p_j \rangle + 2\kappa' \kappa_j \langle f', i(f - p_j) \rangle \\ &= \kappa^2 \langle i f', p'_j \rangle + \kappa' \langle f', p'_j \rangle \\ &= 2\kappa^2 \kappa_j \langle f', f - p_j \rangle + 2\kappa' \kappa_j (\langle f', i(f - p_j) \rangle - \frac{1}{\kappa}) \\ &= 2\kappa^2 \kappa_j \langle f', f - p_j \rangle - \frac{2\kappa' \kappa_j}{\kappa} + 2\kappa' \kappa_j \langle f', i(f - p_j) \rangle \end{aligned}$$

Thus, we have $\kappa' \kappa_j = 0$. Suppose there exists a j such that $\kappa_j \neq 0$. Then κ must be a nonzero constant which implies f is a circle in \mathbf{C}^* . For each s , and t_j , $f(s) - p_j(t_j) = \langle f(s) - p_j, f' \rangle f' + \langle f(s) - p_j, i f' \rangle i f'$ which implies that $p_2 = \dots = p_n$ equal to a constant p since f is a unit circle. Then (3.5) implies that $\kappa_j = 0$ for all $j = 2, \dots, n$ which is a contradiction to our assumption. Therefore, $\kappa_j = 0$ for all $j = 2, \dots, n$. Now, each z_j is a circle in \mathbf{C}^n and by (3.5), and (3.8), $\langle f', i p'_j \rangle = \langle f', p'_j \rangle = 0, j =$

2, ..., n which implies that for each s and t_j , $f(s) - p_j(t_j) = c_1(s)f'(s) + c_2(s)if'(s)$ for some two functions c_1 and c_2 of s , yielding $p_2(t_2) = \dots = p_n(t_n) = p$ become a constant. Therefore, $\mu_2 = \dots = \mu_n = \frac{\langle f', i(f-p) \rangle}{|f-p|^2} = \mu(s)$. Now, then (2.5) becomes $L(s, t_2, \dots, t_n) = (f-p)(z_2 + \dots + z_n)$. Therefore, from (3.4), we obtain

$$(3.9) \quad |f-p|^2 \kappa(s) + (n-1) \langle f', i(f-p) \rangle = 0$$

By differentiating the equation (3.9) with respect to s , we get

$$\kappa'(s)|f-p|^2 + (n+1)\kappa \langle f', f-p \rangle = 0$$

and then it yields

$$(3.10) \quad |f-p|^2 = \left(\frac{a^2}{\kappa(s)} \right)^{\frac{2}{n+1}}$$

for a real constant a . By substituting (3.10) into (3.9), we get

$$(3.11) \quad \langle if', f-p \rangle = \frac{1}{n-1} a^{\frac{4}{n+1}} \kappa(s)^{\frac{n-1}{n+1}}$$

Let's reparametrize $f(s(x)) = p + x + iy(x)$. Then

$$(3.12) \quad \kappa = \langle f'', if' \rangle = \frac{y''(x)}{(1+y'(x)^2)^{3/2}}, \quad \langle if', f-p \rangle = \frac{y-xy'}{(1+y'(x)^2)^{1/2}}$$

From (3.11) and (3.12), we obtain that

$$(3.13) \quad (y-xy')^{n+1} = c(y'')^{n-1}(1+y'(x)^2)^{2-n}, \quad c = \left(\frac{1}{n-1} \right)^{n+1} a^4$$

If $n=2$, then this equation was completely solved by Bang-Yen Chen in his paper [1] which is that up to dilations and rigid motions, the Lagrangian immersion L is an open portion of the Lagrangian catenoid.

Now, we assume that $n \geq 3$. Put $y_1 = y, y_2 = y'$ and $y_3 = y''$. Then the above equation is equivalent to the system:

$$(3.14) \quad \begin{aligned} y_1' &= y_2, y_2' = y_3, \\ y_3' &= \frac{-(1+y_2^2)^{n-2}((n+1)xy_3(y_1-xy_2)^n + 2cy_3^{n-1}(2-n)(1+y_2^2)^{1-n}y_2y_3)}{c(n-1)y_3^{n-2}} \end{aligned}$$

It follows from Picard's theorem that, for a given initial conditions: $y_3(s_0) > 0, y_2(s_0) = y_2^0, y_1(s_0) = y_1^0$ for any constants y_1^0 and y_2^0 , the initial value problem has a unique solution in some open interval around s_0 . □

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