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Classification of Rectifying Space-Like Submanifolds in Pseudo-Euclidean Spaces

Bang-Yen Chen* and Yun Myung Oh

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ABSTRACT

The notions of rectifying subspaces and of rectifying submanifolds were introduced in [B.-Y. Chen, Int. Electron. J. Geom 9 (2016), no. 2, 1–8]. More precisely, a submanifold in a Euclidean m -space \mathbb{E}^m is called a rectifying submanifold if its position vector field always lies in its rectifying subspace. Several fundamental properties and classification of rectifying submanifolds in Euclidean space were obtained in [B.-Y. Chen, op. cit.].

In this present article, we extend the results in [B.-Y. Chen, op. cit.] to rectifying space-like submanifolds in a pseudo-Euclidean space with arbitrary codimension. In particular, we completely classify all rectifying space-like submanifolds in an arbitrary pseudo-Euclidean space with codimension greater than one.

Keywords: Rectifying submanifold; rectifying subspace; pseudo-Euclidean space; concurrent vector field; space-like submanifold; position vector field.

AMS Subject Classification (2010): Primary: 53C40; Secondary: 53C42.

1. Introduction

Let \mathbb{E}^3 denote the Euclidean 3-space with its inner product $\langle \cdot, \cdot \rangle$. Consider a unit-speed space curve $x : I \rightarrow \mathbb{E}^3$, where $I = (\alpha, \beta)$ is a real interval. Let \mathbf{x} denote the position vector field of x and \mathbf{x}' be denoted by \mathbf{t} .

It is possible, in general, that $\mathbf{t}'(s) = 0$ for some s ; however, we assume that this never happens. Then we can introduce a unique vector field \mathbf{n} and positive function κ so that $\mathbf{t}' = \kappa\mathbf{n}$. We call \mathbf{t}' the *curvature vector field*, \mathbf{n} the *principal normal vector field*, and κ the *curvature* of the curve. Since \mathbf{t} is of constant length, \mathbf{n} is orthogonal to \mathbf{t} . The *binormal vector field* is defined by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, which is a unit vector field orthogonal to both \mathbf{t} and \mathbf{n} . One defines the *torsion* τ by the equation $\mathbf{b}' = -\tau\mathbf{n}$.

The famous Frenet-Serret equations are given by

$$\begin{cases} \mathbf{t}' = & \kappa\mathbf{n} \\ \mathbf{n}' = -\kappa\mathbf{t} & + \tau\mathbf{b} \\ \mathbf{b}' = & -\tau\mathbf{n}. \end{cases} \quad (1.1)$$

At each point of the curve, the planes spanned by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$, and $\{\mathbf{n}, \mathbf{b}\}$ are known as the *osculating plane*, the *rectifying plane*, and the *normal plane*, respectively.

From elementary differential geometry it is well known that a curve in \mathbb{E}^3 lies in a plane if its position vector lies in its osculating plane at each point, and it lies on a sphere if its position vector lies in its normal plane at each point. A curve in the Euclidean 3-space is called a rectifying curve if its position vector field always lies in its rectifying plane (cf. [3]). Rectifying curves have many interesting properties. Such curves have been studied by many authors, see for instance, [1, 3, 10, 9, 13, 14, 15] among many others.

In [6], the first author introduced the notion of rectifying subspaces for Euclidean submanifolds. As a natural extension of rectifying curves, the first author defined the notion of rectifying submanifolds as Euclidean submanifolds whose position vector field always lie in its rectifying subspace [6]. Many fundamental properties of rectifying submanifolds are obtained in [6, 7]. In particular, the first author proved that a Euclidean

submanifold is rectifying if and only if the tangential component of its position vector field is a concurrent vector field. Furthermore, he completely determined rectifying submanifolds in a Euclidean space with arbitrary codimension.

In this article we extend the results of [6] to rectifying space-like submanifolds in a pseudo-Euclidean space with arbitrary codimension as a supplement to [6]. In particular, we completely classify all rectifying space-like submanifolds in an arbitrary pseudo-Euclidean space.

2. Preliminaries

For general references on submanifolds in pseudo-Riemannian manifolds, we refer to [5, 8, 16].

Let \mathbb{E}_i^m denote the pseudo-Euclidean m -space equipped with the canonical pseudo-Euclidean metric g_0 of index i given by

$$g_0 = - \sum_{r=1}^i du_r^2 + \sum_{t=i+1}^m du_t^2, \tag{2.1}$$

where (u_1, \dots, u_m) is a rectangular coordinate system of \mathbb{E}_i^m .

Let $x : M \rightarrow \mathbb{E}_i^m$ be an isometric immersion of a pseudo-Riemannian n -manifold M into \mathbb{E}_i^m . For a point $p \in M$, we denote by $T_p M$ and $T_p^\perp M$ the tangent and the normal spaces at p . There is a natural orthogonal decomposition:

$$T_p \mathbb{E}_i^m = T_p M \oplus T_p^\perp M. \tag{2.2}$$

Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and \mathbb{E}_i^m , respectively. The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.3}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.4}$$

for vector fields X, Y tangent to M and ξ normal to M , where h is the second fundamental form, D the normal connection, and A the shape operator of M .

For a given point $p \in M$, the *first normal space*, of M in \mathbb{E}_i^m , denoted by $\text{Im } h_p$, is the subspace defined by

$$\text{Im } h_p = \text{Span}\{h(X, Y) : X, Y \in T_p M\}. \tag{2.5}$$

For each normal vector ξ at p , the shape operator A_ξ is an endomorphism of $T_p M$. The second fundamental form h and the shape operator A are related by

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle, \tag{2.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on M as well as on the ambient space.

The *equation of Gauss* of M in \mathbb{E}_i^m is given by

$$R(X, Y; Z, W) = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle \tag{2.7}$$

for X, Y, Z, W tangent to M , where R denotes the curvature tensors of M .

The covariant derivative $\bar{\nabla}h$ of h with respect to the connection on $TM \oplus T^\perp M$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.8}$$

The *equation of Codazzi* is

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z). \tag{2.9}$$

It follows from the definition of a rectifying curve $x : I \rightarrow \mathbb{E}^3$ that the position vector field \mathbf{x} of x satisfies

$$\mathbf{x}(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s) \tag{2.10}$$

for some functions λ and μ .

For a curve $x : I \rightarrow \mathbb{E}^3$ with $\kappa(s_0) \neq 0$ at $s_0 \in I$, the first normal space at s_0 is the line spanned by the principal normal vector $\mathbf{n}(s_0)$. Hence, the rectifying plane at s_0 is nothing but the plane orthogonal to the first normal space at s_0 . Therefore, for a submanifold M of \mathbb{E}_i^m and a point $p \in M$, we call the subspace of $T_p\mathbb{E}_i^m$, orthogonal complement to the first normal space $\text{Im } h_p$, the *rectifying space* of M at p (see [6]).

We make the following definition as in [6].

Definition 2.1. A pseudo-Riemannian submanifold M of a pseudo-Euclidean space \mathbb{E}_i^m is called a *rectifying submanifold* if the position vector field \mathbf{x} of M always lies in its rectifying space. In other words, M is a rectifying submanifold if and only if

$$\langle \mathbf{x}(p), \text{Im } h_p \rangle = 0 \quad (2.11)$$

holds at every $p \in M$.

3. Lemmas

A tangent vector v of a pseudo-Riemannian manifold \tilde{M}_i^m is called *space-like* (respectively, *time-like*) if $v = 0$ or $\langle v, v \rangle > 0$ (respectively, $\langle v, v \rangle < 0$). A vector v is called *light-like* or *null* if $v \neq 0$ and $\langle v, v \rangle = 0$.

The *light cone* \mathcal{LC} of \mathbb{E}_i^m is defined by

$$\mathcal{LC} = \{v \in \mathbb{E}_i^m : \langle v, v \rangle = 0\}. \quad (3.1)$$

Let r be a positive number. We put

$$S_i^k(r^2) = \{\mathbf{x} \in \mathbb{E}_i^{k+1} : \langle \mathbf{x}, \mathbf{x} \rangle = r^2\}, \quad i > 0, \quad (3.2)$$

$$H_i^k(-r^2) = \{\mathbf{x} \in \mathbb{E}_{i+1}^{k+1} : \langle \mathbf{x}, \mathbf{x} \rangle = -r^2\}, \quad i > 0, \quad (3.3)$$

$$H^k(c) = \{\mathbf{x} \in \mathbb{E}_1^{k+1} : \langle \mathbf{x}, \mathbf{x} \rangle = -r^2 \text{ and } x_1 > 0\}, \quad (3.4)$$

$S_i^k(r^2)$ (respectively, $H_i^k(-r^2)$) is a pseudo-Riemannian manifold of curvature $1/r^2$ (respectively, $-1/r^2$) with index i . The $S_i^k(r^2)$ (respectively, $H_i^k(-r^2)$) is known as a *pseudo-sphere* (respectively, *pseudo-hyperbolic space*).

The pseudo-Riemannian manifolds \mathbb{E}_i^k , $S_i^k(r^2)$, $H_i^k(-r^2)$ are the standard models of the *indefinite real space forms*. In particular, \mathbb{E}_1^k , $S_1^k(c)$, $H_1^k(c)$ are the standard models of *Lorentzian space forms*.

A submanifold M of \mathbb{E}_i^m is called *space-like* if each tangent vector of M is space-like.

By a *cone in \mathbb{E}_i^m with vertex at the origin $o \in \mathbb{E}_i^m$* we mean a ruled submanifold generated by a family of half lines initiated at o . A submanifold of \mathbb{E}_i^m is called a *conic submanifold* with vertex at o if it is an open portion of a cone with vertex at o .

For a space-like submanifold M of \mathbb{E}_i^m , there exists a natural orthogonal decomposition of the position vector field \mathbf{x} at each point; namely,

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^N, \quad (3.5)$$

where \mathbf{x}^T and \mathbf{x}^N denote the tangential and normal components of \mathbf{x} , respectively.

We put

$$|\mathbf{x}^T|^2 = \langle \mathbf{x}^T, \mathbf{x}^T \rangle, \quad |\mathbf{x}^N|^2 = \langle \mathbf{x}^N, \mathbf{x}^N \rangle.$$

Lemma 3.1. *Let M be a pseudo-Riemannian submanifold of the pseudo-Euclidean space \mathbb{E}_i^m . If the position vector field \mathbf{x} of M in \mathbb{E}_i^m is either space-like or time-like, then $\mathbf{x} = \mathbf{x}^T$ holds identically if and only if M is a conic submanifold with the vertex at the origin.*

Proof. Let M be a pseudo-Riemannian submanifold of \mathbb{E}_i^m . Assume that the position vector field \mathbf{x} of M in \mathbb{E}_i^m is either space-like or time-like. If $\mathbf{x} = \mathbf{x}^T$ holds identically, then $e_1 = \mathbf{x}/|\mathbf{x}|$ is a unit vector field.

Put $\mathbf{x} = \rho e_1$. Then we get

$$\tilde{\nabla}_{e_1} \mathbf{x} = e_1, \quad \tilde{\nabla}_{e_1} \mathbf{x} = (e_1 \rho) e_1 + \rho \tilde{\nabla}_{e_1} e_1. \quad (3.6)$$

Since $\tilde{\nabla}_{e_1} e_1$ is perpendicular to e_1 , we find from (3.6) that $\tilde{\nabla}_{e_1} e_1 = 0$. Therefore the integral curves of e_1 are some open portions of generating lines in \mathbb{E}^m . Moreover, because $\mathbf{x} = \mathbf{x}^T$, the generating lines given by the integral curves of e_1 pass through the origin. Consequently, M is a conic submanifold with the vertex at the origin.

The converse is clear. □

We recall the following definition of concurrent vector fields.

Definition 3.1. A non-trivial vector field C on a Riemannian (or more generally, on a pseudo-Riemannian) manifold M is called a *concurrent vector field* if it satisfies

$$\nabla_X C = X \tag{3.7}$$

for any vector X tangent to M , where ∇ is the Levi-Civita connection of M .

Remark 3.1. Since the position vector field of the pseudo-Euclidean space \mathbb{E}_i^m is a concurrent vector field, it follows that the position vector field \mathbf{x} of any pseudo-Riemannian submanifold M in \mathbb{E}_i^m satisfies

$$\tilde{\nabla}_Z \mathbf{x} = Z \tag{3.8}$$

for any $Z \in TM$, where $\tilde{\nabla}$ is the Levi-Civita connection of \mathbb{E}_i^m .

Lemma 3.2. Let M be a pseudo-Riemannian submanifold of \mathbb{E}_i^m . If the position vector field \mathbf{x} is either space-like or time-like, then the position vector field \mathbf{x} of M satisfies $\mathbf{x} = \mathbf{x}^N$ identically if and only if M lies in one of the following hypersurfaces of \mathbb{E}_i^m :

- (1) a pseudo-sphere $S_i^{m-1}(c^2)$; or
- (2) a pseudo-hyperbolic space $H_{i-1}^{m-1}(-c^2)$ whenever $i > 1$; or
- (3) a hyperbolic space $H^{m-1}(-c^2)$ whenever $i = 1$,

where c is a positive number.

Proof. Let $x : M \rightarrow \mathbb{E}_i^m$ be an isometric immersion of a pseudo-Riemannian n -manifold into \mathbb{E}_i^m with space-like or time-like position vector field. If $\mathbf{x} = \mathbf{x}^N$ holds identically, then we get from (3.8) that

$$Z \langle \mathbf{x}, \mathbf{x} \rangle = 2 \langle \tilde{\nabla}_Z \mathbf{x}, \mathbf{x} \rangle = 2 \langle Z, \mathbf{x}^N \rangle = 0$$

for any $Z \in TM$. Thus M lies in one of the three hypersurfaces of \mathbb{E}_i^m .

The converse is easy to verify. □

In views of Lemma 3.1 and Lemma 3.2 we make the following.

Definition 3.2. A rectifying submanifold M of \mathbb{E}_i^m is called *proper* if its position vector field \mathbf{x} satisfies $\mathbf{x} \neq \mathbf{x}^T$ and $\mathbf{x} \neq \mathbf{x}^N$ at every point on M .

In this article, we are only interested on proper rectifying submanifolds of \mathbb{E}_i^m in views of Lemma 3.1 and Lemma 3.2.

For the proof of our main theorem we also need the following lemma.

Lemma 3.3. Let M be a pseudo-Riemannian submanifold of \mathbb{E}_i^m . If M is proper rectifying, then $\langle \mathbf{x}^N, \mathbf{x}^N \rangle$ is constant on M .

Proof. Let $x : M \rightarrow \mathbb{E}_i^m$ be an isometric immersion of a Riemannian n -manifold into \mathbb{E}_i^m . Consider the orthogonal decomposition

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^N \tag{3.9}$$

of the position vector field \mathbf{x} of M in \mathbb{E}_i^m . It follows from (3.9) and the formula of Gauss and the formula of Weingarten that

$$Z = \tilde{\nabla}_Z \mathbf{x} = \nabla_Z \mathbf{x}^T + h(Z, \mathbf{x}^T) - A_{\mathbf{x}^N} Z + D_Z \mathbf{x}^N \tag{3.10}$$

for any $Z \in TM$. By comparing the normal components in (3.10), we find

$$D_Z \mathbf{x}^N = -h(Z, \mathbf{x}^T). \tag{3.11}$$

Therefore we obtain

$$Z \langle \mathbf{x}^N, \mathbf{x}^N \rangle = 2 \langle D_Z \mathbf{x}^N, \mathbf{x}^N \rangle = - \langle h(Z, \mathbf{x}^T), \mathbf{x} \rangle = 0, \tag{3.12}$$

where we have used (2.11) in Definition 2.1. Since (3.12) holds identically for any $Z \in TM$, we conclude that $\langle \mathbf{x}^N, \mathbf{x}^N \rangle$ is constant on M . □

Remark 3.2. A submanifold M of \mathbb{E}_i^m is called a *T-submanifold* (respectively, *N-submanifold*) if its position vector field \mathbf{x} satisfies $\langle \mathbf{x}^T, \mathbf{x}^T \rangle = \text{constant}$ (respectively, $\langle \mathbf{x}^N, \mathbf{x}^N \rangle = \text{constant}$) (cf. [2, 4]). Obviously, Lemma 3.3 implies that every proper rectifying pseudo-Riemannian submanifold of \mathbb{E}_i^m is an N -submanifold.

4. Characterization of rectifying submanifolds in \mathbb{E}_i^m

The following result provides a very simple characterization of rectifying submanifolds.

Theorem 4.1. *If the position vector field \mathbf{x} of a pseudo-Riemannian submanifold M in \mathbb{E}_i^m satisfies $\mathbf{x}^N \neq 0$, then M is a proper rectifying submanifold if and only if \mathbf{x}^T is a concurrent vector field on M .*

Proof. Let M be a space-like submanifold of \mathbb{E}_i^m . Then (3.10) holds. After comparing the tangential components in (3.10), we obtain

$$A_{\mathbf{x}^N} Z = \nabla_Z \mathbf{x}^T - Z. \quad (4.1)$$

Assume that M is a proper rectifying submanifold of \mathbb{E}_i^m . Then we have $\mathbf{x}^T \neq 0$ and $\mathbf{x}^N \neq 0$. Moreover, it follows from the Definition 2.1 that

$$\langle A_{\mathbf{x}^N} X, Y \rangle = \langle \mathbf{x}, h(X, Y) \rangle = 0 \quad (4.2)$$

for $X, Y \in TM$. Since M is space-like, we find from (4.1) that $A_{\mathbf{x}^N} = 0$. Therefore (3.8) yields

$$\nabla_Z \mathbf{x}^T = Z, \quad (4.3)$$

for any $Z \in TM$. Consequently, \mathbf{x}^T is a concurrent vector field on M .

Conversely, if \mathbf{x}^T is a concurrent vector field on M , then (3.7) and (4.1) give $A_{\mathbf{x}^N} = 0$. Therefore we obtain (4.3). Consequently, M is a proper rectifying submanifold due to $\mathbf{x}^N \neq 0$ by assumption. \square

The next result shows that every proper rectifying space-like submanifold is a warped product.

Theorem 4.2. *Let M be a proper rectifying space-like submanifold M of \mathbb{E}_i^m . Then M is a warped product manifold $I \times_s F$ with warping metric*

$$g = ds^2 + s^2 g_F, \quad (4.4)$$

such that $\mathbf{x}^T = s\partial/\partial s$ and g_F is the metric tensor of a Riemannian manifold F .

Proof. Let M be a proper rectifying space-like submanifold of \mathbb{E}_i^m . Then we have $\mathbf{x}^T \neq 0$ and $\mathbf{x}^N \neq 0$. Thus we may put

$$\mathbf{x}^T = \rho e_1, \quad \rho = |\mathbf{x}^T| > 0, \quad (4.5)$$

where e_1 is a space-like unit vector field. We may extend e_1 to a local orthonormal frame e_1, e_2, \dots, e_n on M .

Obviously, it follows from (4.5) that $\rho = \langle \mathbf{x}, e_1 \rangle$. Thus, by taking the derivative of ρ with respect to e_j for $j = 1, \dots, n$ and using (2.3) and (3.8), we find

$$e_j \rho = \delta_{1j} + \langle \mathbf{x}, h(e_1, e_j) \rangle, \quad (4.6)$$

where $\delta_{ij} = 1$ or 0 depending on $i = j$ or $i \neq j$. Combining (2.11) and (4.6) gives

$$e_1 \rho = 1, \quad e_2 \rho = \dots = e_n \rho = 0.$$

Therefore we get $\rho = \rho(s)$ and $\rho'(s) = 1$, which imply $\rho(s) = s + b$ for some real number b . Hence, after applying a suitable translation on s if necessary, we have $\rho = s$. Therefore, we obtain

$$\mathbf{x}^T = s e_1 = s \frac{\partial}{\partial s}. \quad (4.7)$$

Since M is a proper rectifying space-like submanifold, Theorem 4.1 implies that $\mathbf{x}^T = s e_1$ is a concurrent vector field. Thus we find from (4.3) that

$$e_1 = \nabla_{e_1} \mathbf{x}^T = \nabla_{e_1} s e_1 = e_1 + s \nabla_{e_1} e_1, \quad (4.8)$$

which implies $\nabla_{e_1} e_1 = 0$. Therefore the integral curves of e_1 are geodesics of M . Consequently, the distribution \mathcal{D}^\perp spanned by e_1 is a totally geodesic foliation.

From (4.3) we also find

$$e_i = \nabla_{e_i} \mathbf{x}^T = s \nabla_{e_i} e_1, \quad i = 2, \dots, n, \tag{4.9}$$

which gives

$$\omega_1^j(e_i) = \frac{\delta_{ij}}{s}, \quad i, j = 2, \dots, n. \tag{4.10}$$

We conclude from (4.10) that the distribution \mathcal{D} is integrable whose leaves are totally umbilical hypersurfaces of M . Moreover, it follows from (4.10) that the mean curvature of leaves of \mathcal{D} are given by s^{-1} . Since the leaves of \mathcal{D} are hypersurfaces, it follows that the mean curvature vector field of the leaves of \mathcal{D}_2 is parallel in the normal bundle in M . Therefore the distribution \mathcal{D} is a spherical foliation. Consequently, by applying a result of [12] (or Theorem 4.4 of [5, page 90]) we conclude that M is locally a warped product $I \times_s F$, where F is a Riemannian $(n - 1)$ -manifold. Therefore the metric tensor g of M takes the form (4.4). \square

5. Main result

The main result of this article is the following classification theorem.

Theorem 5.1. *Let M be a proper rectifying space-like submanifold of the pseudo-Euclidean m -space \mathbb{E}_i^m with index $i > 0$. If $\text{codim } M \geq 2$, then one of the following four cases occurs:*

- (a) *There exist a positive number c and local coordinate systems $\{s, u_2, \dots, u_n\}$ on M such that the immersion of M in \mathbb{E}_i^m is given by*

$$\mathbf{x}(s, u_2, \dots, u_n) = \sqrt{s^2 + c^2} Y(s, u_2, \dots, u_n), \tag{5.1}$$

where $Y = Y(s, u_2, \dots, u_n)$ defines a space-like submanifolds of the unit pseudo-sphere $S_i^{m-1}(1) \subset \mathbb{E}_i^m$ such that the induced metric g_Y of Y is given by

$$g_Y = \frac{c^2}{(s^2 + c^2)^2} ds^2 + \frac{s^2}{s^2 + c^2} \sum_{j,k=2}^n g_{jk}(u_2, \dots, u_n) du_j du_k. \tag{5.2}$$

- (b) *There exist local coordinate systems $\{s, u_2, \dots, u_n\}$ on M such that the immersion of M in \mathbb{E}_i^m is given by*

$$\mathbf{x}(s, u_2, \dots, u_n) = sW(s, u_2, \dots, u_n), \quad s \neq 0, \tag{5.3}$$

where $W = W(s, u_2, \dots, u_n)$ lies in the unit pseudo-sphere $S_i^{m-1}(1) \subset \mathbb{E}_i^m$ such that W_s is a light-like normal vector field of M and the induced metric tensor of W is of the following degenerate form:

$$g_W = \sum_{j,k=2}^n g_{jk}(u_2, \dots, u_n) du_j du_k \tag{5.4}$$

with positive definite (g_{jk}) , $j, k = 2, \dots, n$.

- (c) *There exist a positive number c and local coordinate systems $\{s, u_2, \dots, u_n\}$ on M such that the immersion of M in \mathbb{E}_i^m is given by*

$$\mathbf{x}(s, u_2, \dots, u_n) = \sqrt{s^2 - c^2} U(s, u_2, \dots, u_n), \quad s^2 > c^2, \tag{5.5}$$

where $U = U(s, u_2, \dots, u_n)$ lies in the unit pseudo-sphere $S_i^{m-1}(1) \subset \mathbb{E}_i^m$ such that the induced metric g_U of U is given by

$$g_U = \frac{-c^2}{(s^2 - c^2)^2} ds^2 + \frac{s^2}{s^2 - c^2} \sum_{j,k=2}^n g_{jk}(u_2, \dots, u_n) du_j du_k. \tag{5.6}$$

- (d) *There exist a positive number c and local coordinate systems $\{s, u_2, \dots, u_n\}$ on M such that the immersion of M in \mathbb{E}_i^m is given by*

$$\mathbf{x}(s, u_2, \dots, u_n) = \sqrt{c^2 - s^2} V(s, u_2, \dots, u_n), \quad c^2 > s^2, \tag{5.7}$$

where $V = V(s, u_2, \dots, u_n)$ lies in the pseudo-hyperbolic space $H_{i-1}^{m-1}(-1) \subset \mathbb{E}_i^m$ for $i > 1$ (respectively, hyperbolic space $H^{m-1}(-1) \subset \mathbb{E}_1^m$ for $i = 1$) such that the induced metric g_V of V is given by

$$g_V = \frac{c^2}{(c^2 - s^2)^2} ds^2 + \frac{s^2}{c^2 - s^2} \sum_{j,k=2}^n g_{jk}(u_2, \dots, u_n) du_j du_k. \tag{5.8}$$

Conversely, each of the four cases above gives rise to a proper rectifying space-like submanifold of \mathbb{E}_i^m .

Proof. Assume that M is a proper rectifying space-like submanifold of \mathbb{E}_i^m with $m \geq 2 + \dim M$. Then we have $\mathbf{x}^T \neq 0$ and $\mathbf{x}^N \neq 0$. Thus we may put

$$\mathbf{x}^T = \rho e_1, \quad \rho = |\mathbf{x}^T| > 0, \quad (5.9)$$

where e_1 is a space-like unit vector field. We may extend e_1 to a local orthonormal frame e_1, e_2, \dots, e_n on M . Clearly, we have $\langle \mathbf{x}, e_j \rangle = 0$ for $j = 2, \dots, n$.

Define the connection forms $\omega_i^j, i, j = 1, \dots, n$, by

$$\nabla_X e_i = \sum_{j=1}^n \omega_i^j(X) e_j, \quad i = 1, \dots, n, \quad (5.10)$$

where ∇ is the Levi-Civita connection of M .

For $j, k = 2, \dots, n$, we find

$$0 = e_k \langle \mathbf{x}, e_j \rangle = \delta_{jk} + \langle \mathbf{x}, \nabla_{e_k} e_j \rangle + \langle \mathbf{x}, h(e_j, e_k) \rangle = \delta_{jk} + \langle \mathbf{x}, \nabla_{e_k} e_j \rangle, \quad (5.11)$$

where we have applied (2.11) from Definition 2.1, (2.3) and (3.8).

Since $h(X, Y)$ is symmetric in X and Y , we derive from (5.10) and (5.11) that

$$\omega_j^1(e_k) = \omega_k^1(e_j), \quad j, k = 2, \dots, n. \quad (5.12)$$

It follows from (5.10), (5.12) and the Frobenius theorem that the distribution \mathcal{D} spanned by e_2, \dots, e_n is an integrable distribution.

On the other hand, the distribution $\mathcal{D}^\perp = \text{Span}\{e_1\}$ is also integrable since it is of rank one. Therefore, there exists a local coordinate system $\{s, u_2, \dots, u_n\}$ on M such that

$$e_1 = \frac{\partial}{\partial s} \quad \text{and} \quad \mathcal{D} = \text{Span} \left\{ \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n} \right\}.$$

Obviously, it follows from (5.9) that $\rho = \langle \mathbf{x}, e_1 \rangle$. Now, by taking the derivative of ρ with respect to e_j for $j = 1, \dots, n$ and using (2.3) and (3.8), we find

$$e_j \rho = \delta_{1j} + \langle \mathbf{x}, h(e_1, e_j) \rangle. \quad (5.13)$$

After combining (2.11) and (5.13) we find $e_1 \rho = 1$ and $e_2 \rho = \dots = e_n \rho = 0$. Therefore we have

$$\rho = \rho(s), \quad \rho'(s) = 1$$

which imply

$$\rho(s) = s + b. \quad (5.14)$$

for some real number b . Consequently, after applying a suitable translation on s if necessary, we obtain $\rho = s$. Consequently, (5.9) implies that the position vector field satisfies

$$\mathbf{x} = s e_1 + \mathbf{x}^N. \quad (5.15)$$

Moreover, since M is a proper rectifying submanifold, Lemma 3.3 implies that $\langle \mathbf{x}^N, \mathbf{x}^N \rangle$ is constant on M . Therefore we find

$$\langle \mathbf{x}, \mathbf{x} \rangle = \begin{cases} s^2 + c^2, & \text{if } \langle \mathbf{x}^N, \mathbf{x}^N \rangle > 0, \\ s^2, & \text{if } \langle \mathbf{x}^N, \mathbf{x}^N \rangle = 0, \\ s^2 - c^2, & \text{if } \langle \mathbf{x}^N, \mathbf{x}^N \rangle < 0, \end{cases} \quad (5.16)$$

where c is a positive number.

Now, we divide the proof of the theorem into three cases.

Case (1): $\langle \mathbf{x}, \mathbf{x} \rangle = s^2 + c^2$ with $c > 0$. In this case, we may put

$$\mathbf{x}(s, u_2, \dots, u_n) = \sqrt{s^2 + c^2} Y(s, u_2, \dots, u_n), \quad (5.17)$$

for some \mathbb{E}_i^m -valued function $Y = Y(s, u_2, \dots, u_n)$ satisfying $\langle Y, Y \rangle = 1$. Therefore the image of Y lies in the pseudo-sphere $S_i^{m-1}(1) \subset \mathbb{E}_i^{m-1}$. It follows from (5.17) that

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial s} &= \frac{s}{\sqrt{s^2 + c^2}} Y + \sqrt{s^2 + c^2} Y_s, \\ \frac{\partial \mathbf{x}}{\partial u_j} &= \sqrt{s^2 + c^2} Y_{u_j}, \quad j = 2, \dots, n. \end{aligned} \tag{5.18}$$

Using (5.18) together with the fact that $e_1 = \partial \mathbf{x} / \partial s$ is a unit vector field orthogonal to the distribution \mathcal{D} , we derive that

$$\langle Y_s, Y_s \rangle = \frac{c^2}{(s^2 + c^2)^2}, \quad \langle Y_s, Y_{u_j} \rangle = 0, \quad j = 2, \dots, n. \tag{5.19}$$

Therefore the metric tensor g_Y of Y induced from $S_i^{m-1}(1)$ takes the following form:

$$g_Y = \frac{c^2}{(s^2 + c^2)^2} ds^2 + \frac{s^2}{s^2 + c^2} \sum_{j,k=2}^n g_{jk}(s, u_2, \dots, u_n) du_j du_k, \tag{5.20}$$

where (g_{jk}) is positive definite. In particular, (5.17) and (5.20) show that the submanifold defined by Y is also space-like.

Now, by applying (5.18) and (5.20) we know that the metric tensor g of M is of the form:

$$g = ds^2 + s^2 \sum_{j,k=2}^n g_{jk}(s, u_2, \dots, u_n) du_j du_k. \tag{5.21}$$

After a straight-forward long computation we find from (5.21) that the Levi-Civita connection of M satisfies

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} &= 0, \\ \nabla_{\frac{\partial}{\partial u_j}} \frac{\partial}{\partial s} &= \frac{1}{s} \frac{\partial}{\partial u_j} + \frac{1}{2} \sum_{k=2}^n \left(\sum_{t=2}^n g^{kt} \frac{\partial g_{jt}}{\partial s} \right) \frac{\partial}{\partial u_k}, \quad j = 2, \dots, n, \end{aligned} \tag{5.22}$$

where (g^{jk}) is the inverse matrix of (g_{ij}) . Because M is a proper rectifying space-like submanifold of \mathbb{E}_i^m , it follows from Theorem 4.1 that

$$\nabla_{\frac{\partial}{\partial u_j}} \mathbf{x}^T = \frac{\partial}{\partial u_j}, \quad j = 2, \dots, n. \tag{5.23}$$

Therefore, after applying (4.7), (5.22) and (5.23) we obtain

$$\sum_{t=2}^n g^{kt} \frac{\partial g_{jt}}{\partial s} = 0, \quad j, k = 2, \dots, n. \tag{5.24}$$

Because (g^{jk}) is positive definite, system (5.24) implies

$$\frac{\partial g_{jk}}{\partial s} = 0, \quad j, t = 2, \dots, n.$$

Therefore (5.31) must take the form of (5.4). Consequently, (5.20) reduces to (5.2).

Conversely, let us consider a space-like submanifold M of \mathbb{E}_i^m defined by (5.1) satisfying $\langle Y, Y \rangle = 1$ such that the metric tensor g_Y is given by (5.2). Then we obtain (5.18) and (5.19) from (5.1). It follows from (5.2), (5.18) and (5.19) that the metric tensor g of M is given by

$$g = ds^2 + s^2 \sum_{j,k=2}^n g_{jk}(u_2, \dots, u_n) du_j du_k. \tag{5.25}$$

Now, it is straight-forward to verify from (5.25) that the Levi-Civita connection of M satisfies

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0, \quad \nabla_{\frac{\partial}{\partial u_j}} \frac{\partial}{\partial s} = \frac{1}{s} \frac{\partial}{\partial u_j}, \quad j = 2, \dots, n. \tag{5.26}$$

Since $\langle Y, Y \rangle = 1$, (5.1) implies $\langle \mathbf{x}, Y_{u_j} \rangle = 0$ for $j = 2, \dots, n$. Thus we find from (5.18) that

$$\langle \mathbf{x}, \mathbf{x}_{u_j} \rangle = 0, \quad j = 2, \dots, n. \quad (5.27)$$

Therefore, we obtain $\mathbf{x}^T = s \frac{\partial}{\partial s}$. Now, by applying (5.26) it is easy to verify that \mathbf{x}^T is a concurrent vector field on M . Moreover, it is direct to show that the normal component of \mathbf{x} is given by

$$\mathbf{x}^N = \frac{c^2}{\sqrt{s^2 + c^2}} Y - s \sqrt{s^2 + c^2} Y_s,$$

which is always non-zero everywhere on M . Consequently, the immersion defined by case (a) gives rise to a proper rectifying space-like submanifold of \mathbb{E}_i^m .

Case (2): $\langle \mathbf{x}, \mathbf{x} \rangle = s^2, s \neq 0$. In this case, \mathbf{x}^N is a light-like normal vector field of M .

We put

$$\mathbf{x}(s, u_2, \dots, u_n) = s W(s, u_2, \dots, u_n), \quad s \neq 0, \quad (5.28)$$

for some \mathbb{E}_i^m -valued function $W = W(s, u_2, \dots, u_n)$ satisfying $\langle W, W \rangle = 1$. Therefore the image of W lies in the pseudo-sphere $S_i^{m-1}(1) \subset \mathbb{E}_i^{m-1}$.

It follows from (5.28) that

$$\frac{\partial \mathbf{x}}{\partial s} = W + s W_s, \quad \frac{\partial \mathbf{x}}{\partial u_j} = s W_{u_j}, \quad j = 2, \dots, n. \quad (5.29)$$

Using (5.29), $\langle W, W \rangle = 1$ and the fact that $e_1 = \partial \mathbf{x} / \partial s$ is a unit vector field orthogonal to the distribution \mathcal{D} , we derive that

$$\langle W_s, W_s \rangle = 0, \quad \langle W_s, W_{u_j} \rangle = 0, \quad j = 2, \dots, n. \quad (5.30)$$

If we put $g_{jk} = \langle W_{u_j}, W_{u_k} \rangle$, then it follows from (5.29) and (5.30) that the metric tensor g_W of W is a generate one given by

$$g_W = \sum_{j,k=2}^n g_{ij}(s, u_2, \dots, u_n) du_j du_k. \quad (5.31)$$

Then it follows from (5.28) and (5.31) that the induced metric g of M is given by

$$g = ds^2 + s^2 \sum_{j,k=2}^n g_{jk}(s, u_2, \dots, u_n) du_j du_k. \quad (5.32)$$

Since M is a proper rectifying space-like submanifold of \mathbb{E}_i^m , it follows from Theorem 4.1 that \mathbf{x}^T is a concurrent vector field. Therefore, we may apply the same argument as in Case (1) to conclude that $\partial g_{jk} / \partial s = 0$ for $j, k = 2, \dots, n$. Therefore (5.31) must take the form of (5.4).

Conversely, let us consider an immersion $x : M \rightarrow \mathbb{E}_i^m$ of a Riemannian n -manifold M into \mathbb{E}_i^m given by

$$\mathbf{x}(s, u_2, \dots, u_n) = s W(s, u_2, \dots, u_n), \quad \langle W, W \rangle = 1, \quad s \neq 0, \quad (5.33)$$

such that W_s is a light-like normal vector field and the metric tensor of W is of the following degenerate form:

$$g_W = \sum_{j,k=2}^n g_{jk}(u_2, \dots, u_n) du_j du_k \quad (5.34)$$

with positive definite matrix $(g_{jk}), j, k = 2, \dots, n$. Then it follows from (5.33) and (5.34) that the induced metric g of M is given by

$$g = ds^2 + s^2 \sum_{j,k=2}^n g_{jk}(u_2, \dots, u_n) du_j du_k. \quad (5.35)$$

From (5.34) we get

$$\mathbf{x}_s = W + s W_s, \quad \mathbf{x}_{u_j} = s W_{u_j}, \quad j = 2, \dots, n. \quad (5.36)$$

Thus we find from (5.33) and (5.36) that

$$\mathbf{x} = s\mathbf{x}_s - s^2W_s. \quad (5.37)$$

Because W_s is a light-like normal vector field and \mathbf{x}_s is tangent to M , we obtain from (5.37) that

$$\mathbf{x}^T = s\mathbf{x}_s \text{ and } \mathbf{x}^N = -s^2W_s \neq 0. \quad (5.38)$$

Now, we may derive from (5.35) and (5.38) as before that \mathbf{x}^T is a concurrent vector field on M . Consequently, M is a rectifying space-like submanifold of \mathbb{E}_i^m according to Theorem 4.1. This gives Case (b) of the theorem.

Case (3): $\langle \mathbf{x}, \mathbf{x} \rangle = s^2 - c^2 \neq 0$. By applying a method similar to Case (1), we will obtain either Case (c) or Case (d) according to $s^2 > c^2$ or $s^2 < c^2$, respectively. \square

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