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ESTIMATES OF LIFE SPAN OF SOLUTIONS OF A CAUCHY PROBLEM

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Abstract: In this paper we get estimates of life span of a Cauchy problem

$$\begin{aligned}u_t(x, t) &= \Delta u(x, t) + u(x, t)^p, & x \in R^n, t > 0, \\u(x, 0) &= \lambda \phi(x), & x \in R^n\end{aligned}$$

in terms of the positive constant parameter λ when $\phi(x) \in L^q$ is a nonnegative bounded continuous function in R^n but not identically zero, where q is large enough. The technique we used in this paper is the Comparison Principle.

1. Introduction

In this paper we consider the Cauchy problem

$$\begin{aligned}u_t(x, t) &= \Delta u(x, t) + u(x, t)^p, & x \in R^n, t > 0, \\u(x, 0) &= \lambda \phi(x), & x \in R^n,\end{aligned}\tag{1}$$

where $\Delta = \sum_{i=1}^n (\frac{\partial^2}{\partial x_i^2})$ is the Laplace operator, $p > 1$, $\phi \in L^q$ is a nonnegative bounded continuous function in R^n but not identically zero, where q is large enough, and λ is a positive constant parameter. It is well known that there exists an $T_\lambda > 0$ such that (1) possesses a unique classical solution $u(x, t, \lambda)$ in $[0, T_\lambda)$, i.e., $u(x, t, \lambda) \in C^{2,1}(R^n \times (0, T_\lambda)) \cap C(R^n \times [0, T_\lambda))$ is bounded in $[0, T']$ for any $T' < T_\lambda$ and $\|u(\cdot, t, \lambda)\|_{L^\infty} \rightarrow \infty$ when $t \rightarrow T_\lambda$ if T_λ is finite. We call T_λ the life span of the solution $u(x, t, \lambda)$ and say that $u(x, t, \lambda)$ blows up in

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finite time if $T_\lambda < \infty$.

Since Fujita’s classic work [1], (1) has been studied extensively in a lot of directions. For stability and instability results, the interested readers are referred to [2] for a survey and some new developments; You can also refer to [4] for some other related results. Motivated by a paper of Lee and Ni [5], we are concerned with asymptotic behavior of the life span T_λ as $\lambda \rightarrow \infty$ or $\lambda \rightarrow 0$. The following was proved in [5]:

Theorem 1.1. (1) $T_\lambda \sim \lambda^{-(p-1)}$ as $\lambda \rightarrow \infty$; i.e., there exist positive constants C_1 and C_2 such that $C_1\lambda^{-(p-1)} \leq T_\lambda \leq C_2\lambda^{-(p-1)}$ for large λ .

(2) If $\liminf_{|x| \rightarrow \infty} \phi(x) > 0$, then $T_\lambda < \infty$ for any $\lambda > 0$ and $T_\lambda \sim \lambda^{-(p-1)}$ as $\lambda \rightarrow 0$.

In [3] they improved the 1.1. The following was the result.

Theorem 1.2. (1) $\lim_{\lambda \rightarrow \infty} T_\lambda \lambda^{p-1} = \frac{1}{p-1} \|\phi\|_{L^\infty}^{-(p-1)}$.

(2) If $\lim_{|x| \rightarrow \infty} \phi(x) = \phi_\infty > 0$, then

$$\lim_{\lambda \rightarrow 0} T_\lambda \lambda^{p-1} = \frac{1}{p-1} \phi_\infty^{-(p-1)}.$$

In this research, we prove other estimates of the life span of (1) when we assumed $\phi \in L^q$ for large enough q using the Comparison Principle.

2. Main Results

Theorem 2.1. (1) $T_\lambda \geq \frac{1}{2^{p-1}(p-1)} \lambda^{1-p} \|\phi\|_{L^q}^{-(p-1)}$ for some large enough q . So, $\lim_{\lambda \rightarrow 0} T_\lambda = \infty$.

(2) $T_\lambda \leq \frac{1}{2^{(p-1)}} \lambda^{1-p} \|\phi\|_{L^r}^{-(p-1)}$ for some large enough r . So, $\lim_{\lambda \rightarrow \infty} T_\lambda = 0$.

Proof. (1) Let $v_\lambda(x, t) = \frac{\lambda^{-1}}{2} u(\lambda^{\frac{1-p}{2}} x, \lambda^{1-p} t)$. Then v_λ satisfies

$$\begin{aligned} \frac{\partial v_\lambda(x, t)}{\partial t} &= \frac{\lambda^{-1}}{2} \lambda^{1-p} \frac{\partial u(z, \tau)}{\partial \tau} \\ &= \frac{\lambda^{-p}}{2} \frac{\partial u(z, \tau)}{\partial \tau} \\ &= \frac{\lambda^{-p}}{2} [\Delta_z u(z, \tau) + u^p(z, \tau)] \\ &= \frac{\lambda^{-p}}{2} \Delta_z u(z, \tau) + \frac{\lambda^{-p}}{2} u^p(z, \tau) \\ &= \frac{\lambda^{-1}}{2} \lambda^{1-p} \Delta_z u(z, \tau) + \frac{\lambda^{-p}}{2} u^p(z, \tau) \\ &= \Delta v_\lambda(x, t) + 2^{p-1} v_\lambda^p(x, t), x \in R^n \times (0, \tilde{T}_\lambda), \\ v_\lambda(x, 0) &= \frac{1}{2} \phi(\lambda^{\frac{1-p}{2}} x), x \in R^n, \end{aligned}$$

where $\tilde{T}_\lambda = \lambda^{p-1}T_\lambda$ is the life span of v_λ , $z = \lambda^{\frac{1-p}{2}}x$ and $\tau = \lambda_{1-p}t$. Since $\lim_{s \rightarrow \infty} \|\phi\|_{L^s} = \|\phi\|_{L^\infty}$, there is a large enough q such that $\frac{1}{2} \|\phi\|_{L^\infty} < \|\phi\|_{L^q}$. Then we may consider the following ordinary differential equation.

$$\begin{aligned} \frac{dv(t)}{dt} &= 2^{p-1}v(t)^p, t > 0, \\ v(0) &= \|\phi\|_{L^q}. \end{aligned}$$

The ordinary differential equation implies that

$$\frac{1}{v^p}dv = 2^{p-1}dt,$$

so

$$\int v^{-p}dv = \int 2^{p-1}dt + C,$$

so

$$\frac{1}{1-p}v^{1-p} = 2^{p-1}t + C.$$

But, by applying the initial condition $v(0) = \|\phi\|_{L^q}$, we have

$$C = \frac{1}{1-p} \|\phi\|_{L^q}^{1-p},$$

and so

$$\frac{1}{1-p}v^{1-p} = 2^{p-1}t + \frac{1}{1-p} \|\phi\|_{L^q}^{1-p}.$$

Therefore,

$$v^{p-1} = \frac{1}{2^{p-1}(1-p)t + \|\phi\|_{L^q}^{1-p}},$$

and so the life span of v is $T = \frac{1}{2^{p-1}(p-1)} \|\phi\|_{L^q}^{-(p-1)}$.

By the Comparison Principle (see [6]), we have

$$\tilde{T}_\lambda = \lambda^{p-1}T_\lambda \geq \frac{1}{2^{p-1}(p-1)} \|\phi\|_{L^q}^{-(p-1)},$$

and so we conclude (1).

(2) Let $v_\lambda(x, t) = 2\lambda^{-1}u(\lambda^{\frac{1-p}{2}}x, \lambda^{1-p}t)$. Then v_λ satisfies

$$\begin{aligned} \frac{\partial v_\lambda(x, t)}{\partial t} &= 2\lambda^{-1}\lambda^{1-p}\frac{\partial u(z, \tau)}{\partial \tau} \\ &= 2\lambda^{-p}\frac{\partial u(z, \tau)}{\partial \tau} \\ &= 2\lambda^{-p}[\Delta_z u(z, \tau) + u^p(z, \tau)] \\ &= 2\lambda^{-p}\Delta_z u(z, \tau) + 2\lambda^{-p}u^p(z, \tau) \\ &= 2\lambda^{-1}\lambda^{1-p}\Delta_z u(z, \tau) + \frac{2}{2^p}u^p(z, \tau) \\ &= 2\lambda^{-1}\Delta u(\lambda^{\frac{1-p}{2}}x, \lambda^{1-p}t) + \frac{1}{2^{p-1}}v_\lambda^p(x, t) \\ &= \Delta v_\lambda(x, t) + \frac{1}{2^{p-1}}v_\lambda(x, t), x \in R^n \times (0, \tilde{T}_\lambda), \\ v_\lambda(x, 0) &= 2\phi(\lambda^{\frac{1-p}{2}}x), x \in R^n, \end{aligned}$$

where $\tilde{T}_\lambda = \lambda^{p-1}T_\lambda$ is the life span of v_λ , $z = \lambda^{\frac{1-p}{2}}x$ and $\tau = \lambda^{1-p}t$. Since $\lim_{s \rightarrow \infty} \|\phi\|_{L^s} = \|\phi\|_{L^\infty}$, there is a large enough r such that

$$2\|\phi\|_{L^\infty} > \|\phi\|_{L^r}.$$

Then we may consider the following ordinary differential equation.

$$\begin{aligned} \frac{dv(t)}{dt} &= \frac{1}{2^{p-1}}v(t)^p, t > 0, \\ v(0) &= \|\phi\|_{L^r}. \end{aligned}$$

The ordinary differential equation implies that

$$\frac{1}{v^p}dv = \frac{1}{2^{p-1}}dt,$$

so

$$\int \frac{1}{v^p}dv = \int \frac{1}{2^{p-1}}dt + C,$$

so

$$\frac{1}{1-p}v^{1-p} = \frac{1}{2^{p-1}}t + C.$$

But, by applying the initial condition $v(0) = \|\phi\|_{L^r}$, we have

$$C = \frac{1}{1-p} \|\phi\|_{L^r}^{1-p},$$

and so

$$\frac{1}{1-p}v^{1-p} = \frac{1}{2^{p-1}}t + \frac{1}{1-p} \|\phi\|_{L^r}^{1-p}.$$

Therefore,

$$v^{1-p} = \frac{1-p}{2^{p-1}}t + \|\phi\|_{L^r}^{1-p},$$

so

$$v^{p-1} = \frac{1}{\frac{1-p}{2^{p-1}}t + \|\phi\|_{L^r}^{1-p}},$$

and so the life span of v is $T = \frac{2^{p-1}}{p-1} \|\phi\|_{L^r}^{-(p-1)}$.

By the Comparison Principle (see [6]), we have

$$\tilde{T}_\lambda = \lambda^{p-1} T_\lambda \leq \frac{2^{p-1}}{p-1} \|\phi\|_{L^r}^{-(p-1)},$$

and so we conclude (2).

References

- [1] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **16** (1966), 105-113.
- [2] C. Gui, W.-M. Ni, and X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in R^n , *Comm. Pure Appl. Math.*, **XLV** (1992), 1153-1181.
- [3] C. Gui, X. Wang, Life span of solutions of the Cauchy problem for a semilinear heat equation, *Journal of Differential Equations*, **115** (1995), 166-172.
- [4] H. Levine, The role of critical exponents on blow-up theorems, *SIAM Rev.*, **32** (1990), 262-288.
- [5] T.-Y. Lee and W.-M. Ni, Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem, *Trans. Amer. Math. Soc.*, **333** (1992), 365-371.
- [6] M.H. Protter, H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.
- [7] X. Wang, On Cauchy problems for reaction-diffusion equations, *Trans. Amer. Math. Soc.*, **337** (1993), 549-590.

