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## A GENERAL ELLIPTIC NONLINEAR SYSTEM OF TWO FUNCTIONS WITH APPLICATION

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**Abstract:** The purpose of this paper is to give a sufficient condition for the existence and nonexistence of positive solutions to a rather general type of elliptic system of the Dirichlet problem on the bounded domain  $\Omega$  in  $R^n$ . Also considered are the effects of perturbations on the coexistence state and uniqueness. The techniques used in this paper are upper-lower solutions, eigenvalues of operators, maximum principles and spectrum estimates. The arguments also rely on some detailed properties for the solution of logistic equations. These results yield an algebraically computable criterion for the positive coexistence of competing species of animals in many biological models.

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**Key Words:** general elliptic system, positive solution

### 1. Introduction

The most general type of elliptic interacting system of two functions with homogeneous boundary condition is

$$\begin{cases} \Delta u + g(u, v) = 0, \\ \Delta v + h(u, v) = 0, \\ (u, v)|_{\partial\Omega} = (0, 0), \end{cases} \quad \text{in } \Omega, \quad (1)$$

where we assume that the  $C^2$  functions  $g$  and  $h$  are relative growth rates satisfying the following so-called growth rate conditions:

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$$(G_1) \quad g_v < 0, g_{uu}(u, v) < 0, g_{uv}(u, v) < 0, h_u < 0, h_{uv}(u, v) < 0, h_{vv}(u, v) < 0,$$

$$(G_2) \quad g(0, v) \geq 0, h(u, 0) \geq 0,$$

(G<sub>3</sub>) There exist constants  $c_0 > 0$  such that  $g(u, 0) < 0, g_u(u, 0) < 0, h(0, v) < 0, h_v(0, v) < 0$  for  $u > c_0$  and  $v > c_0$ .

The goal of this paper is to answer the following questions about positive solutions of (1).

**Problem 1.** What are the sufficient conditions for existence of positive solutions?

**Problem 2.** What are the sufficient conditions for nonexistence of positive solutions?

**Problem 3.** What is the effect of perturbation for existence and uniqueness?

## 2. Preliminaries

In this section we state some preliminary results which will be useful for our later arguments.

**Definition 2.1.** (Upper and Lower solutions)

The vector functions  $(\bar{u}^1, \dots, \bar{u}^N), (\underline{u}^1, \dots, \underline{u}^N)$  form an upper/lower solution pair for the system

$$\begin{cases} \Delta u^i + g^i(u^1, \dots, u^N) = 0 & \text{in } \Omega \\ u^i = 0 & \text{on } \partial\Omega \end{cases}$$

if for  $i = 1, \dots, N$

$$\begin{cases} \Delta \bar{u}^i + g^i(u^1, \dots, u^{i-1}, \bar{u}^i, u^{i+1}, \dots, u^N) \leq 0 \\ \Delta \underline{u}^i + g^i(u^1, \dots, u^{i-1}, \underline{u}^i, u^{i+1}, \dots, u^N) \geq 0 \\ \text{in } \Omega \text{ for } \underline{u}^j \leq u^j \leq \bar{u}^j, j \neq i, \end{cases}$$

and

$$\begin{aligned} \underline{u}^i &\leq \bar{u}^i && \text{on } \Omega \\ \underline{u}^i &\leq 0 \leq \bar{u}^i && \text{on } \partial\Omega. \end{aligned}$$

**Lemma 2.1.** If  $g^i$  in the Definition 2.1 are in  $C^1$  and the system admits an upper/lower solution pair  $(\underline{u}^1, \dots, \underline{u}^N), (\bar{u}^1, \dots, \bar{u}^N)$ , then there is a solution of the system in 2.1 with  $\underline{u}^i \leq u^i \leq \bar{u}^i$  in  $\Omega$ . If

$$\begin{aligned} \Delta \bar{u}^i + g^i(\bar{u}^1, \dots, \bar{u}^N) &\neq 0, \\ \Delta \underline{u}^i + g^i(\underline{u}^1, \dots, \underline{u}^N) &\neq 0 \end{aligned}$$

in  $\Omega$  for  $i = 1, \dots, N$ , then  $\underline{u}^i < u^i < \bar{u}^i$  in  $\Omega$ .

**Lemma 2.2.** (The first eigenvalue)

$$\begin{cases} -\Delta u + q(x)u = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2}$$

where  $q(x)$  is a smooth function from  $\Omega$  to  $R$  and  $\Omega$  is a bounded domain in  $R^n$ .

(A) The first eigenvalue  $\lambda_1(q)$  of (2), denoted by simply  $\lambda_1$  when  $q \equiv 0$ , is simple with a positive eigenfunction.

(B) If  $q_1(x) < q_2(x)$  for all  $x \in \Omega$ , then  $\lambda_1(q_1) < \lambda_1(q_2)$ .

(C) (Variational Characterization of the first eigenvalue)

$$\lambda_1(q) = \min_{\phi \in W_0^1(\Omega), \phi \neq 0} \frac{\int_{\Omega} (|\nabla \phi|^2 + q\phi^2) dx}{\int_{\Omega} \phi^2 dx}$$

**Lemma 2.3.** (Maximum Principles)

$$Lu = \sum_{i,j=1}^n a_{ij}(x)D_{ij}u + \sum_{i=1}^n a_i(x)D_iu + a(x)u = f(x) \text{ in } \Omega,$$

where  $\Omega$  is a bounded domain in  $R^n$ .

(M1)  $\partial\Omega \in C^{2,\alpha}$  ( $0 < \alpha < 1$ )

(M2)  $|a_{ij}(x)|_{\alpha}, |a_i(x)|_{\alpha}, |a(x)|_{\alpha} \leq M$  ( $i, j = 1, \dots, n$ )

(M3)  $L$  is uniformly elliptic in  $\bar{\Omega}$ , with ellipticity constant  $\gamma$ , i.e., for every  $x \in \bar{\Omega}$  and every real vector  $\xi = (\xi_1, \dots, \xi_n)$

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_{i=1}^n |\xi_i|^2.$$

Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  be a solution of  $Lu \geq 0$  ( $Lu \leq 0$ ) in  $\Omega$ .

(A) If  $a(x) \equiv 0$ , then  $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$  ( $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$ ).

(B) If  $a(x) \equiv 0$  and  $u$  attains its maximum (minimum) at an interior point of  $\Omega$ , then  $u$  is identically a constant in  $\Omega$ .

**Lemma 2.4.**

$$\begin{cases} \Delta u + uf(u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where  $f$  is a decreasing  $C^1$  function such that there exists  $c_0 > 0$  such that  $f(u) \leq 0$  for  $u \geq c_0$  and  $\Omega$  is a bounded domain in  $R^n$ .

(1) If  $f(0) > \lambda_1$ , then the above equation has a unique positive solution, where  $\lambda_1$  is the first eigenvalue of  $-\Delta$  with homogeneous boundary condition. We denote this unique positive solution as  $\theta_f$ .

(2) If  $f(0) \leq \lambda_1$ , then the above equation does not have any positive solution.

### 3. Existence, Nonexistence

We consider the system (1) with conditions  $(G_1)$ ,  $(G_2)$  and  $(G_3)$ .

**Theorem 3.1.** (A) If  $g_u(0, c_0) > \lambda_1$  and  $h_v(c_0, 0) > \lambda_1$ , then (1) has a solution  $(u, v)$  with  $u > 0, v > 0$ .

Any solution  $(u, v)$  of (1) with  $u > 0, v > 0$  in  $\Omega$  satisfies

$$\begin{aligned}\theta_{g_u(\cdot, c_0)} &< u \leq c_0 \\ \theta_{h_v(c_0, \cdot)} &< v \leq c_0.\end{aligned}$$

(B) If  $g_u(0, 0) \leq \lambda_1$  or  $h_v(0, 0) \leq \lambda_1$ , then (1) does not have any positive solutions.

*Proof.* (A) By the result in [8], there is a positive solution  $(u_1, v_1)$  to

$$\begin{aligned}\Delta u + ug_u(u, v) &= 0, \\ \Delta v + vh_v(u, v) &= 0 \text{ in } \Omega, \\ (u, v)|_{\partial\Omega} &= (0, 0).\end{aligned}$$

But, by the Mean Value Theorem, there is  $\tilde{u}$  such that  $0 \leq \tilde{u} \leq u_1$  and  $g(u_1, v_1) - g(0, v_1) = u_1 g_u(\tilde{u}, v_1)$ , so by the monotonicity of  $g_u$  and  $(G_2)$ ,

$$\begin{aligned}&\Delta u_1 + g(u_1, v_1) \\ &\geq \Delta u_1 + g(u_1, v_1) - g(0, v_1) \\ &= \Delta u_1 + u_1 g_u(\tilde{u}, v_1) \\ &\geq \Delta u_1 + u_1 g_u(u_1, v_1) \\ &= 0.\end{aligned}$$

Similarly,

$$\Delta v_1 + h(u_1, v_1) \geq 0.$$

Hence,  $(u_1, v_1)$  is a subsolution to (1).

But, by  $(G_1)$  and  $(G_3)$ , any  $(M, M)$  with large enough  $M > 0$  is a supersolution to (1).

Therefore, by Lemma 2.1, there is a positive solution to (1).

Suppose  $(u, v)$  is a solution to (1) with  $u > 0, v > 0$ .

Then by the Mean Value Theorem, there is  $\bar{u}$  such that  $0 \leq \bar{u} \leq u$  and  $g(u, v) - g(0, v) = u g_u(\bar{u}, v)$ , so by the monotonicity of  $g_u$ ,

$$\begin{aligned}&\Delta u + ug_u(u, v) \\ &\leq \Delta u + ug_u(\bar{u}, v) \\ &= \Delta u + g(u, v) - g(0, v) \\ &\leq \Delta u + g(u, v) \\ &= 0.\end{aligned}$$

Similarly,

$$\Delta v + v h_v(u, v) \leq 0.$$

Hence,  $(u, v)$  is a supersolution to

$$\begin{aligned} \Delta u + u g_u(u, v) &= 0, \\ \Delta v + v h_v(u, v) &= 0 \text{ in } \Omega, \\ (u, v)|_{\partial\Omega} &= (0, 0). \end{aligned}$$

For sufficiently large  $n \in N$ , by the monotonicity of  $g_u$ ,

$$\begin{aligned} &\Delta\left[\frac{1}{n}\theta_{g_u(\cdot, c_0)}\right] + \frac{1}{n}\theta_{g_u(\cdot, c_0)}g_u\left(\frac{1}{n}\theta_{g_u(\cdot, c_0)}, \frac{1}{n}\theta_{h_v(c_0, \cdot)}\right) \\ &= \frac{1}{n}\left[\Delta[\theta_{g_u(\cdot, c_0)}] + \theta_{g_u(\cdot, c_0)}g_u\left(\frac{1}{n}\theta_{g_u(\cdot, c_0)}, \frac{1}{n}\theta_{h_v(c_0, \cdot)}\right)\right] \\ &\geq \frac{1}{n}\left[\Delta[\theta_{g_u(\cdot, c_0)}] + \theta_{g_u(\cdot, c_0)}g_u(\theta_{g_u(\cdot, c_0)}, c_0)\right] \\ &= 0, \end{aligned}$$

and similarly,

$$\Delta\left[\frac{1}{n}\theta_{h_v(c_0, \cdot)}\right] + \frac{1}{n}\theta_{h_v(c_0, \cdot)}h_v\left(\frac{1}{n}\theta_{g_u(\cdot, c_0)}, \frac{1}{n}\theta_{h_v(c_0, \cdot)}\right) \geq 0,$$

so  $(\frac{1}{n}\theta_{g_u(\cdot, c_0)}, \frac{1}{n}\theta_{h_v(c_0, \cdot)})$  is a subsolution to

$$\begin{aligned} \Delta u + u g_u(u, v) &= 0, \\ \Delta v + v h_v(u, v) &= 0, \\ (u, v)|_{\partial\Omega} &= (0, 0). \end{aligned}$$

Hence, by Lemma 2.1, there is a positive solution  $(u_1, v_1)$  to

$$\begin{cases} \Delta u + u g_u(u, v) = 0, \\ \Delta v + v h_v(u, v) = 0, \\ (u, v)|_{\partial\Omega} = (0, 0), \end{cases}$$

with  $u_1 \leq u, v_1 \leq v$ .

By the solution estimates in [8], we conclude

$$\begin{aligned} \theta_{g_u(\cdot, c_0)} &< u \\ \theta_{h_v(c_0, \cdot)} &< v. \end{aligned} \tag{3}$$

We prove  $u(x) \leq c_0, v(x) \leq c_0$  for  $x \in \bar{\Omega}$ .

If  $u(x) > c_0$  at some  $x \in \Omega$ , then by the continuity of  $u$ , there is  $r > 0$  such that  $u(y) > c_0$  for all  $y \in B_r(x) = \{y \in R^n \mid \|y - x\| < r\}$ , so by the monotonicity of  $g$ ,

$$\Delta u = -g(u, v) > -g(c_0, 0) \geq 0 \text{ on } B_r(x),$$

which contradicts the Maximum Principles.

Hence, we conclude

$$u \leq c_0, v \leq c_0. \tag{4}$$

By (3) and (4), we have the desired inequalities.

(B) Assume  $g_u(0, 0) \leq \lambda_1$ . The other cases are proved similarly. Suppose  $(\bar{u}, \bar{v})$  is a positive solution to (1). Then, by the same way in (A), we have a positive solution  $(u_1, v_1)$  to

$$\begin{cases} \Delta u + u g_u(u, v) = 0, \\ \Delta v + v h_v(u, v) = 0, \\ (u, v)|_{\partial\Omega} = (0, 0). \end{cases}$$

Hence, we again obtain a contradiction to Lemma 2.4 by the same reason as in [8]. Therefore, there is no positive solution to (1).

#### 4. Uniqueness with Perturbation

We consider the model

$$\begin{cases} \Delta u + g(u, v) = 0 \\ \Delta v + h(u, v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega, \tag{5}$$

where  $\Omega$  is a smooth, bounded domain in  $R^n$  and (P1)  $g, h \in C_B^2$ , where  $C_B^2$  is the set of all two variables functions  $f(u, v)$  such that all the first-order partial derivatives of  $f$  are decreasing, and all the second-order partial derivatives of  $f$  are bounded and continuous,

(P2) there are  $c_0, c_1 > 0$  such that  $g_u(u, 0) < 0$  for  $u > c_0$  and  $h_v(0, v) < 0$  for  $v > c_1$ .

The following theorem is our main result about the perturbation of uniqueness.

**Theorem 4.1.** *Suppose*

(A)  $\lambda_1(-g_u(0, \theta_{h_v(0, \cdot)})) < 0, \lambda_1(-h_v(\theta_{g_u(\cdot, 0)}, 0)) < 0$ , where in general,  $\lambda_1(q)$  is the smallest eigenvalue of  $-\Delta + q$  with homogeneous boundary conditions, denoted simply by  $\lambda_1$  when  $q \equiv 0$ ,

(B) (5) has a unique coexistence state  $(u, v)$ ,

(C) the Frechet derivative of (5) at  $(u, v)$  is invertible.

Then there is a neighborhood  $V$  of  $(g, h)$  in  $[C_B^2(R^2)]^2$  such that if  $(\bar{g}, \bar{h}) \in V$ , then (5) with  $(\bar{g}, \bar{h})$  has a unique positive solution.

*Proof.* Since the Frechet derivative of (5) at  $(u, v)$  is invertible, then by the Implicit Function Theorem there is a neighborhood  $V$  of  $(g, h)$  in  $C_B^2$  and a neighborhood  $W$  of  $(u, v)$  in  $[C_0^{2,\alpha}(\bar{\Omega})]^2$  such that for all  $(\bar{g}, \bar{h}) \in V$  there is a unique positive solution  $(\bar{u}, \bar{v}) \in W$  of (5). Thus, the local uniqueness of the solution is guaranteed.

To prove global uniqueness, suppose that the conclusion of Theorem 4.1 is false.

Then, there are sequences  $(g_n, h_n, u_n, v_n), (g_n, h_n, \bar{u}_n, \bar{v}_n)$  in  $V \times [C_0^{2,\alpha}(\bar{\Omega})]^2$  such that  $(u_n, v_n)$  and  $(\bar{u}_n, \bar{v}_n)$  are positive solutions of (5) with  $(g_n, h_n)$  and  $(u_n, v_n) \neq (\bar{u}_n, \bar{v}_n)$  and  $(g_n, h_n) \rightarrow (g, h)$ . By Schauder's estimate in elliptic theory, the convergence of  $(g_n, h_n)$ , and the solution estimate in the proof of Theorem 3.1, there are constants  $k_1 > 0, k_2 > 0, k_3 > 0, k_4 > 0$  such that

$$\begin{aligned} |u_n|_{2,\alpha} &\leq k_1 [|g_n(u_n, v_n)|_\alpha + \sup_{x \in \bar{\Omega}} (u_n(x))] \leq k_1(k_3 + c_0), \\ |v_n|_{2,\alpha} &\leq k_2 [|h_n(u_n, v_n)|_\alpha + \sup_{x \in \bar{\Omega}} (v_n(x))] \leq k_2(k_4 + c_1) \end{aligned}$$

for all  $n = 1, 2, \dots$ , and so

we conclude that  $|u_n|_{2,\alpha}$  and  $|v_n|_{2,\alpha}$  are uniformly bounded. Therefore, there are uniformly convergent subsequences of  $u_n$  and  $v_n$ , which again will be denoted by  $u_n$  and  $v_n$ .

Thus, let

$$\begin{aligned} (u_n, v_n) &\rightarrow (\hat{u}, \hat{v}), \\ (\bar{u}_n, \bar{v}_n) &\rightarrow (\bar{u}, \bar{v}) \end{aligned}$$

Then  $(\hat{u}, \hat{v}), (\bar{u}, \bar{v}) \in (C^{2,\alpha})^2$  are also solutions of (5) with  $(g, h)$ . We claim that  $\hat{u} > 0, \hat{v} > 0, \bar{u} > 0, \bar{v} > 0$ .

By the same proof as in Section 3, for each  $n = 1, 2, \dots$ , there is a positive solution  $(\tilde{u}_n, \tilde{v}_n)$  to

$$\begin{cases} \Delta u + u(g_n)_u(u, v) = 0 \\ \Delta v + v(h_n)_v(u, v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \end{cases} \quad \text{in } \Omega, \tag{6}$$

such that  $\tilde{u}_n \leq u_n, \tilde{v}_n \leq v_n$ .

By exactly the same proof as in [7], there are uniformly convergent subsequences of  $\tilde{u}_n$  and  $\tilde{v}_n$ , which again will be denoted by  $\tilde{u}_n$  and  $\tilde{v}_n$ ,  $\tilde{u}_n \rightarrow \tilde{u}, \tilde{v}_n \rightarrow \tilde{v}$ ,  $(\tilde{u}, \tilde{v})$  is a solution to (6) with  $(g_u, h_v)$ , and  $\tilde{u} > 0, \tilde{v} > 0$ .

But, since  $\tilde{u}_n \leq u_n, \tilde{v}_n \leq v_n, 0 < \tilde{u} \leq \hat{u}, 0 < \tilde{v} \leq \hat{v}$ .

By the same procedure with the sequence  $(\bar{u}_n, \bar{v}_n)$ , we also have  $\bar{u} > 0, \bar{v} > 0$ .

In conclusion, both  $(\hat{u}, \hat{v})$  and  $(\bar{u}, \bar{v})$  are positive solutions to (6) with  $(g, h)$ . But, by condition (B),  $\hat{u} = \bar{u} = u, \hat{v} = \bar{v} = v$ . This is a contradiction to the Implicit Function Theorem, since  $(u_n, v_n) \neq (\bar{u}_n, \bar{v}_n)$ .

### 5. Uniqueness with Perturbation of Region

We consider the model

$$\begin{cases} \Delta u + g(u, v) = 0 \\ \Delta v + h(u, v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega, \tag{7}$$

where  $\Omega$  is a smooth, bounded domain in  $R^n$  and  $g, h \in C_B^2$ , where  $C_B^2$  is the set of all two variables functions  $f(u, v)$  such that all the first-order partial derivatives of



$f$  are decreasing, and all the second-order partial derivatives of  $f$  are bounded and continuous.

The following Theorem is the main result.

**Theorem 5.1.** *Suppose*

- (A)  $\Gamma$  is a closed, bounded, convex region in  $[C_B^1]^2$  such that for all  $(g, h) \in \Gamma$ ,  $\lambda_1(-g_u(0, \theta_{h_v(0, \cdot)})) < 0$  and  $\lambda_1(-h_v(\theta_{g_u(\cdot, 0)}, 0)) < 0$ ,
- (B) there exist  $c_0 > 0$  and  $c_1 > 0$  such that for all  $(g, h) \in \Gamma$ ,  $g_u(u, 0) < 0$  for  $u > c_0$  and  $h_v(0, v) < 0$  for  $v > c_1$ ,
- (C) (7) has a unique positive solution for every  $(g, h) \in \partial_L \Gamma$ , where  $\partial_L \Gamma = \{(\lambda_h, h) \in \Gamma \mid \text{for any fixed } h, \lambda_h = \inf\{\|g\| \mid (g, h) \in \Gamma\}\}$ ,
- (D) for all  $(g, h) \in \Gamma$ , the Fréchet derivative of (7) at every positive solution to (7) is invertible.

Then for all  $(g, h) \in \Gamma$ , (7) has a unique positive solution. Furthermore, there is an open set  $W$  in  $[C_B^1]^2$  such that  $\Gamma \subseteq W$  and for every  $(g, h) \in W$ , (7) has a unique positive solution.

Theorem 5.1 goes even further than Theorem 4.1. Theorem 5.1 states uniqueness in the whole region of  $(g, h)$  whenever we have uniqueness on the left boundary and invertibility of the linearized operator at any particular solution inside the domain.

*Proof.* For each fixed  $h$ , consider  $(g, h) \in \partial_L \Gamma$  and  $(\bar{g}, h) \in \Gamma$ . We need to show that for all  $0 \leq t \leq 1$ , (7) with  $(1 - t)(g, h) + t(\bar{g}, h)$  has a unique positive solution. Since (7) with  $(g, h)$  has a unique positive solution  $(u, v)$  and the Frechet derivative of (7) at  $(u, v)$  is invertible, Theorem 4.1 implies that there is an open neighborhood  $V$  of  $(g, h)$  in  $(C_B^1)^2$  such that if  $(g_0, h_0) \in V$ , then (7) with  $(g_0, h_0)$  has a unique positive solution. Let  $\lambda_s = \sup\{0 \leq \lambda \leq 1 \mid (7) \text{ with } (1 - t)(g, h) + t(\bar{g}, h) \text{ has a unique coexistence state for } 0 \leq t\lambda\}$ . We need to show that  $\lambda_s = 1$ . Suppose  $\lambda_s < 1$ . From the definition of  $\lambda_s$ , there is a sequence  $\{\lambda_n\}$  such that  $\lambda_n \rightarrow \lambda_s^-$  and there is a sequence  $(u_n, v_n)$  of the unique positive solutions of (7) with  $(1 - \lambda_n)(g, h) + \lambda_n(\bar{g}, h)$ . Then by elliptic theory, there is  $(u_0, v_0)$  such that  $(u_n, v_n)$  converges to  $(u_0, v_0)$  uniformly and  $(u_0, v_0)$  is a solution of (7) with  $(1 - \lambda_s)(g, h) + \lambda_s(\bar{g}, h)$ . We claim that both  $u_0$  and  $v_0$  are positive.

By the same proof as in the Section 3, for each  $n = 1, 2, \dots$ , there is a positive solution  $(\tilde{u}_n, \tilde{v}_n)$  to

$$\begin{cases} \Delta u + u[(1 - \lambda_n)g_u + \lambda_n \bar{g}_u](u, v) = 0 \\ \Delta v + v[(1 - \lambda_n)h_v + \lambda_n \bar{h}_v](u, v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \end{cases} \quad \text{in } \Omega, \tag{8}$$

such that  $\tilde{u}_n \leq u_n, \tilde{v}_n \leq v_n$ .

By exactly the same proof as in [7], there are uniformly convergent subsequences of  $\tilde{u}_n$  and  $\tilde{v}_n$ , which again will be denoted by  $\tilde{u}_n$  and  $\tilde{v}_n$ ,  $\tilde{u}_n \rightarrow \tilde{u}, \tilde{v}_n \rightarrow \tilde{v}$ ,  $(\tilde{u}, \tilde{v})$  is a

solution to (8) with  $(1 - \lambda_s)(g_u, h_v) + \lambda_s(\bar{g}_u, \bar{h}_v)$ , and  $\tilde{u} > 0, \tilde{v} > 0$ .

But, since  $\tilde{u}_n \leq u_n, \tilde{v}_n \leq v_n$ , it follows that  $0 < \tilde{u} \leq u_0, 0 < \tilde{v} \leq v_0$ .

We claim that (7) has a unique coexistence state with  $(1 - \lambda_s)(g, h) + \lambda_s(\bar{g}, \bar{h})$ . In fact, if not, assume that  $(\bar{u}_0, \bar{v}_0) \neq (u_0, v_0)$  is another coexistence state. By the Implicit Function Theorem, there exists  $0 < \tilde{a} < \lambda_s$  and very close to  $\lambda_s$  such that (7) with  $(1 - \tilde{a})(g, h) + \tilde{a}(\bar{g}, \bar{h})$  has a coexistence state very close to  $(\bar{u}_0, \bar{v}_0)$ , which means that (7) with  $(1 - \tilde{a})(g, h) + \tilde{a}(\bar{g}, \bar{h})$  has more than one coexistence state. This is a contradiction to the definition of  $\lambda_s$ . But, since (7) with  $(1 - \lambda_s)(g, h) + \lambda_s(\bar{g}, \bar{h})$  has a unique coexistence state and the Frechet derivative is invertible, Theorem 4.1 implies that  $\lambda_s$  can not be as defined. Therefore, for each  $(g, h) \in \Gamma$ , (7) with  $(g, h)$  has a unique coexistence state  $(u, v)$ . Furthermore, by the assumption, for each  $(g, h) \in \Gamma$ , the Frechet derivative of (7) with  $(g, h)$  at the unique solution  $(u, v)$  is invertible. Hence, Theorem 4.1 concluded that there is an open neighborhood  $V_{(g,h)}$  of  $(g, h)$  in  $(C_B^1)^2$  such that if  $(\bar{g}, \bar{h}) \in V_{(g,h)}$ , then (7) with  $(\bar{g}, \bar{h})$  has a unique coexistence state. Let  $W = \bigcup_{(g,h) \in \Gamma} V_{(g,h)}$ . Then  $W$  is an open set in  $(C_B^1)^2$  such that  $\Gamma \subseteq W$  and for each  $(\bar{g}, \bar{h}) \in W$ , (7) with  $(\bar{g}, \bar{h})$  has a unique coexistence state.

## 6. Application

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple competition model, commonly known as the Lotka-Volterra competition model. This system describes the competitive interaction of two species residing in the same environment in the following manner:

$$\begin{cases} u_t(x, t) = \Delta u(x, t) + u(x, t)(a - bu(x, t) - cv(x, t)) \\ v_t(x, t) = \Delta v(x, t) + v(x, t)(d - fv(x, t) - eu(x, t)) \\ u(x, t)|_{\partial\Omega} = v(x, t)|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega \times R^+, \quad (9)$$

where  $\Omega$  is a bounded domain in  $R^n$ . Here,  $u(x, t)$  and  $v(x, t)$  designate the population densities of the two competing species. The positive constant coefficients in this system represent growth rates ( $a$  and  $d$ ), self-limitation rates ( $b$  and  $f$ ) and competition rates ( $c$  and  $e$ ). Furthermore, we assume that both species are not residing on the boundary of  $\Omega$ .

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (9) (see [1], [2], [3], [4], [5]). The positive steady state solution is simply the positive solution to the time-independent system

$$\begin{cases} \Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\ \Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 \\ u(x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = 0. \end{cases} \quad \text{in } \Omega, \quad (10)$$

One of the important initial results for the time-independent Lotka-Volterra model was obtained by Cosner and Lazer. In 1984, they published the following sufficient conditions for the existence and uniqueness of a positive steady state solution to (10):

**Theorem 6.1.** (in [4])

Suppose

(A)  $a > \lambda_1 + \frac{cd}{f}$ ,  $d > \lambda_1 + \frac{ae}{b}$ , where  $\lambda_1$  is the smallest eigenvalue of  $-\Delta$  with homogeneous boundary conditions,

(B)  $4bf > \frac{fc^2}{b} \sup_{x \in \Omega} [\frac{\omega_a(x)}{\omega_{d-\frac{ae}{b}}(x)}] + 2ce + \frac{be^2}{f} \sup_{x \in \Omega} [\frac{\omega_d(x)}{\omega_{a-\frac{cd}{f}}(x)}]$ , where  $\omega_M(x)$  for  $M > 0$

is the unique positive solution to the logistic equation as mentioned in the next section.

Then (2) has a unique positive solution.

Cosner and Lazer's theorem implies that if the self-reproduction and self-limitation rates are relatively large, and the competition rates are relatively small, then there is a unique positive steady state solution to (10). In other words, the two species will coexist indefinitely at unique population densities.

In 1989, Cantrell and Cosner extended these results by proving that the reproduction and self-limitation rates may vary within bounds without losing the uniqueness result, given certain conditions. Biologically, Cantrell and Cosner's theorem suggests that two species can relax ecologically and maintain a coexistence state. Their primary result is given below:

**Theorem 6.2.** (in [3])

If  $a = d > \lambda_1$ ,  $b = f = 1$ , and  $0 < c, e < 1$ , then there is a neighborhood  $V$  of  $(a, a)$  such that if  $(a_0, d_0) \in V$ , then (10) with  $(a, d) = (a_0, d_0)$  has a unique positive solution.

In Theorem 6.2, the condition  $0 < c, e < 1$  biologically implies that the competition rates of both species must be relatively small. This condition plays an important role in the proof of Cantrell and Cosner's theorem by implying the invertibility of the Frechet derivative (linearization) of (10) at a fixed reproduction rate  $(a, a)$ .

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