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Classifying Pretzel Links Obtained by Strong Fusion

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CLASSIFYING PRETZEL LINKS OBTAINED BY STRONG FUSION

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Abstract. A link is a collection of circles embedded into 3-dimensional space. Pretzel links are an important family of links which comprises those links that fit a general form that includes many of the most common links. The strong fusion of a link joins two components of the link via a band and adds an unknotted circle about the band [4]; this naturally arises in the study of concordance and has been used to model biological phenomena such as site-specific recombination in DNA [2]. Here we present a complete and original classification of those pretzel links which can be obtained by strong fusion. The primary tools we depend on are linking number and a dichromatic resolution of the link in which we conceive of the link as being colored with two colors and resolve crossings in such a way that respects those colors. Solving the classification problem in a number of subcases gives the general result.

1. Introduction and Background

Knot theory is an area of mathematics that studies properties of mathematical knots and links. A mathematical knot is a closed, non-self intersecting curve typically thought of as living in 3 dimensional space; a link is a collection of such curves. For instance, in Figure 1 we have a diagram representing the link that we refer to as L6a1. In the naming convention, 6 denotes the number of crossings in the diagram and 1 is an arbitrary index to distinguish the link from other links with 6 crossings.

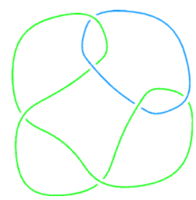


Figure 1. The link L6a1

Links are considered equivalent up to *ambient isotopy*, which means we may deform the link without passing any arcs through crossings. Additionally, we may *orient* knots and links by tracing the direction of a strand.

The origins of knot theory trace to Lord Kelvin in the late 19th century who had proposed that atoms were knotted vortices in the supposed ether [1]. This inspired attempts to tabulate knots and distinguish knots that cannot be manipulated into one another. Ultimately, atoms were found to be different from the proposed knot model. However, mathematicians remained interested in the problem of knots, particularly as connections between knots and other mathematical structures were discovered.

While knot theory developed as an area of pure mathematics, in recent decades, notable applications have been discovered. For instance, the action of site-specific recombinases acting on DNA molecules may be modeled as an operation on links [2]. In particular, this operation, called fusion, merges two components of a knot or a link together with an oriented band. Fusion has been well-studied in the past [2], and links up to 7 crossings that are the result of fusion have been tabulated.

In this project we are interested in strong fusion, a variant of fusion that is of interest to mathematicians. Strong fusion is performed by merging two components of a link with an oriented band and then add an unknotted ($L0a1$) component about the fusion band as in Figure 2. Since fusing two components reduces the total number of components by one, adding the unknotted component around the fusion band preserves the total number of components, making the operation mathematically interesting. The unknotted component is known as the *distinguished component* of strong fusion.

For example, we can obtain the link L7a3 by performing a strong fusion on the link mL4a1.

While fusion has been well-studied in the past, less is known about strong fusion. Note that it is not always obvious when a link is the result of strong fusion, as we consider links up to ambient isotopy and the strong fusion band may be hidden. In the past, my colleagues and I completed an original tabulation of links up to nine

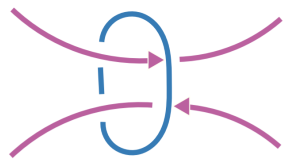


FIGURE 2. The strong fusion band

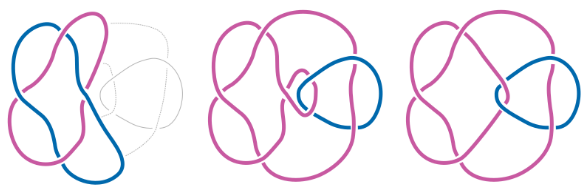


FIGURE 3. L7a3 as a strong fusion of mL4a1

crossings that are the result of strong fusion. However, this tabulation is for a finite collection of links.

For this project, we focus our attention on a infinite family of links called pretzel links. Pretzel links are composed of a series of a series of half twists (called towers) joined at the ends. A pretzel link is denoted by $P(q_1, q_2, ..., q_m)$, where the q_i's indicate the number of half twists in each tower. See Figure 4a. Many links can be represented as pretzel links, making the study of this particular family of links useful.

(A) The pretzel link $P(q_1, \cdots, q_m)$

(B) The pretzel link $P(-3, 5, 7)$

FIGURE 4. Pretzel links

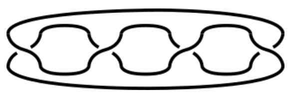
Our main finding is a theorem that allows us to determine exactly when a pretzel link is the result of strong fusion. Notice the theorem is an "if and only if" statement, meaning that it is giving both *necessary* and *sufficient* conditions for when a pretzel link is the result of strong fusion.

Main Theorem. *A pretzel link $P(q_1, q_2, ..., q_n)$ is the result of strong fusion if and only if it is one of the following forms (up to permutation):*

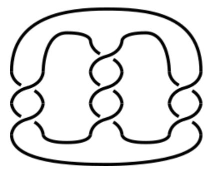
(1) *a trivial link with at least two components,*
(2) *m is odd, the only even values are $q_j = q_k = \pm 2$, and each $p_i = \pm 1$ for all $j < i < k$,*

(3) *m is even, the only even values are $q_j = -q_k = \pm 2$, and each $p_i = \pm 1$ for all $j < i < k$.*

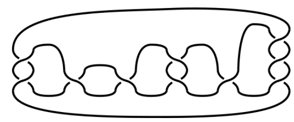
For example, the following links are pretzel links in each of the three categories and thus each of these links is the result of strong fusion. To "see" the strong fusion, one must manipulate the diagrams until one has an unknotted component around a fusion band, but the theorem guarantees this can be accomplished for these links.



(A) $P(-1,1,1,-1)$ satisfies (1).



(B) $P(2,3,2)$ satisfies (2).



(C) $P(2,-1,-1,-2,1,3)$ satisfies (3).

In mathematics, we prove theorems through strict deductive reasoning, building upon other known results. To derive our theorem, our main tools are three invariants. An invariant is a property that remains unchanged under some kind of transformation. For us, we are interested in properties of links that remain unchanged under strong fusion, which allows us to identify when a link is the result of strong fusion.

These invariants are introduced in Section 2. Then in section 3 we use the invariants to determine when a pretzel link is the result of strong fusion and exactly what form such a pretzel link will be in. By dividing up pretzel links into three cases based on the number of even towers that they possess, we can then apply the three tests to completely classify all pretzel links that are the result of strong fusion.

2. Tests for Strong Fusion

We employ three tests involving components, linking number, and dichromatic resolution in order to identify when a link is not the result of strong fusion. Suppose the C is the distinguished component of a link S obtained from strong fusion on a link L. We note some immediate proprieties of component C.

2.1. **Unknotted Component.**

If a link is the result of strong fusion, it must have an unknot as one of its components. This follows directly from the definition of strong fusion, which adds an unknotted component around the fusion band.

Proposition 1. *Let C denote the distinguished component of a link S obtained from strong fusion on link L. Then C is unknotted.*

Proof. That C is unknotted follows immediately from the definition of strong fusion. □

2.2. **Linking Number.**

The *linking number*, denoted $lk(L_1, L_2)$, of two components L_1 and L_2 of a link describes the amount of times the components wind about each other, counted with sign [6]. If a link is the result of strong fusion, the unknotted component must have linking number zero with any other component of the link.

Proposition 2. *Let C denote the distinguished component of a link S obtained from strong fusion on link L. Then C has linking number zero with the other components of L.*

Proof. Let $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m$. Assume without loss of generality that components L_{m-1} and L_m were fused under the strong fusion to obtain L'. In particular, we have $L' = L_1 \sqcup L_2 \sqcup \cdots \sqcup L'_{m-1} \sqcup C$ where L'_{m-1} is the result of fusing L_{m-1} and L_m. To calculate the linking number between C and other components of L', consider the disk D^2 bounded by C. Since $L_1, ..., L_{m-2}$ do not intersect D^2, we have $lk(L_k, C) = 0$ for $1 \leq k \leq m-2$. Moreover, since L'_{m-1} traverses D^2 twice, each time with opposite orientation, we have $lk(L'_{m-1}, C) = 1 - 1 = 0$. □

This is enough to rule out several links as not being the result of strong fusion, but it is not sufficient. For instance, L9a38 has an unknotted component that has linking number zero with all other components, but the link is not the result of strong fusion. To see this, we introduce another test.

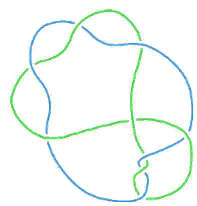


FIGURE 6. L9a38

2.3. **Dichromatic Link Invariant.** The third test involves a process known as *resolving crossings*. We switch any crossing within a link to any of the other two possibilities in Figure 7, and repeating this process with the resulting links creates a *resolution tree*. If a link is the result of strong fusion, the resolution of crossings that do not involve the unknotted component will result in links of a certain form.

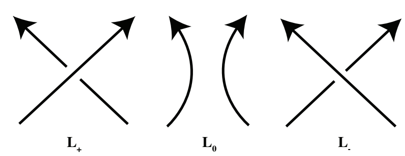


FIGURE 7. Resolution moves

A 1-trivial dichromatic link is a link with at least two components including a single distinguished unknotted circle. Thus, links obtained by strong fusion may be regarded as 1-trivial dichromatic links where the circle added from the strong fusion is the distinguished component. Hoste and Kidwell introduced a skein invariant for such links [3].

Theorem 2.1. *Let $\mathcal{L}^C$ denote the set of isotopy classes of 1-trivial dichromatic links. Then there is a unique map*

$$\Omega : \mathcal{L}^C \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}, \{[E]\}]$$

satisfying

$$v^{-1}\Omega(L_+) - v\Omega(L_-) = z\Omega(L_0)$$

and

$$\Omega(E) = [E]$$

where E is an elementary link $a^{n_1}\#a_{n_2}\#\cdots\#a^{n_k}$ and a^l is the link depicted in Figure 8.

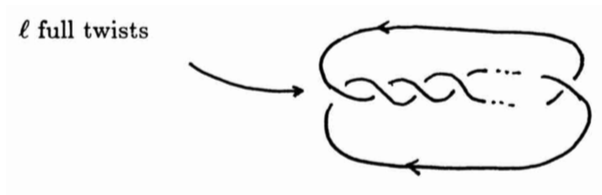


FIGURE 8. The link a^l

Note that the trivial two-component link is also elementary and denoted by 1. Then every elementary link can be considered as a 1-trivial dichromatic link and connected sum between the distinguished components.

Hoste and Kidwell also provided an important restriction on which elementary links may appear in the resolution of the crossings of a link. This restriction involves wrap(L), defined for a 1-trivial dichromatic link to be the minimal geometric intersection number of any disk spanning the distinguished component with the rest of the link.

Theorem 2.2. *Let L be a 1-trivial link with*

$$\Omega(L) = \Sigma\sigma_r(v_j, z_j)[a^{k_{r,1}}\#...\#a^{k_{r,n_r}}].$$

Then

$$max_r\{|k_{r,1}| + \cdots + |k_{r,n_r}|\} \leq wrap(L)$$

where the quantities are equal mod 2.

The next proposition follows from the theorems proven by Hoste and Kidwell, as Kaiser observed [5].

Proposition 3. *Suppose L is a strong fusion. Then $wrap(L) = 2$ and $k_{r,1} + \cdots + k_{r,n_r} = 0$, as linking number is preserved, and thus*

$$\Omega(L) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}, \{[1], [a\#a^{-1}]\}]$$

This gives a way to determine when a link is not the result of strong fusion. In particular, resolve crossings in a 1-trivial dichromatic link L that do not involve the distinguished component. If one obtains an elementary link other than [1] or $[a\#a^{-1}]$, then L is not the result of strong fusion.

Example: Consider the link in Figure 9. While three of the four resolution paths result in elementary links of the form [1] and $[a\#a^{-1}]$, the leftmost resolution gives the elementary link $a\#a^{-1}\#a^{-1}\#a$, indicating that this link cannot be the result of strong fusion.

Given these three tests, we are now equipped to identify pretzel links that are the result of strong fusion.

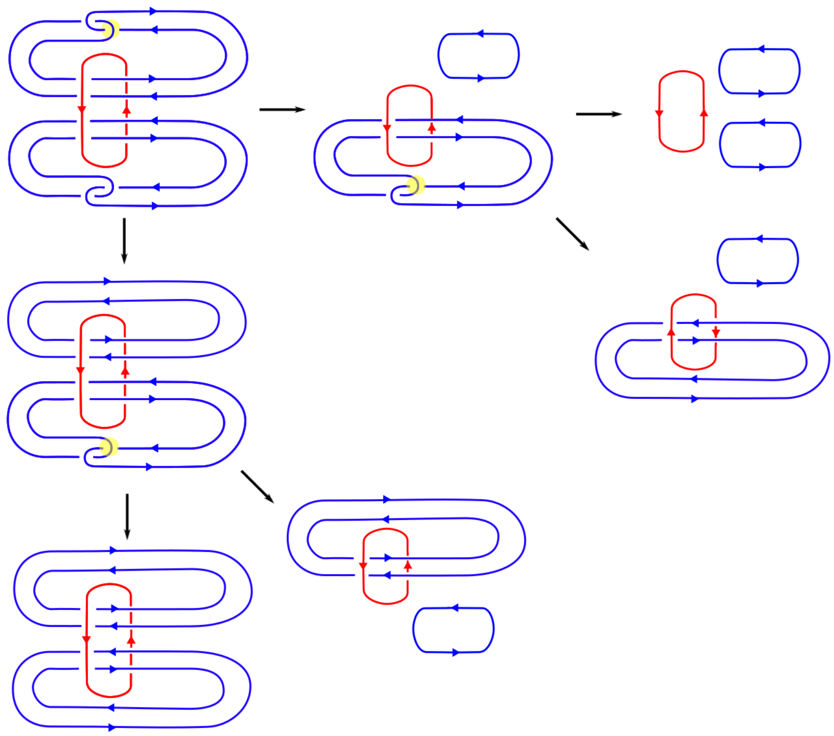


Figure 9. Dichromatic resolution

3. Pretzel Links

Let $P(q_1, q_2, ..., q_m)$ denote the pretzel link comprising m towers of nonzero $q_i \in \mathbb{Z}$ half twists as in Figure 4a. We prove the main theorem by applying the three tests introduced in the previous section in conjunction with the following properties of pretzel links.

Main Theorem. *A pretzel link $P(q_1, q_2, ..., q_n)$ is the result of strong fusion if and only if it is one of the following forms (up to permutation):*

1. *a trivial link with at least two components,*
2. *m is odd, the only even values are $q_j = q_k = \pm 2$, and each $p_i = \pm 1$ for all $j < i < k$,*
3. *m is even, the only even values are $q_j = -q_k = \pm 2$, and each $p_i = \pm 1$ for all $j < i < k$.*

Of relevance are the following propositions which will be used to prove which pretzel links are the result of strong fusion [6].

Lemma 1 (Cycling property of pretzel links). Let $P(q_1, ..., q_{n-1}, q_n)$ be a pretzel link. Then

$$P(q_1, ..., q_{n-1}, q_n) = P(q_n, q_1, ..., q_{n-1})$$

Lemma 2 (Moveability of ± 1 towers). The pretzel link $P(q_1, \cdots, q_i, \cdots, q_n)$ with $q_i = \pm 1$ is positive-equivalent to $P(q_i, q_1, \cdots, \hat{q}_i, \cdots, q_n,$ where $\hat{q}_i$ indicates an omission of the tower with q_i crossings.

Let us denote components in the k^{th} tower in the following manner:

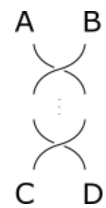


FIGURE 10. Components of a pretzel tower

We will call A and C the left strands of the tower, and B and D are the right strands of the tower. A and B are also the top strands of the tower, while B and D are the bottom strands of the tower. It is of use to note the following properties of towers:

Lemma 3. Given an odd tower, the A and D strands will be part of the same component, and the B and C strands will be part of the same component. Given an even tower, the A and C (right) strands will be part of the same component, and the B and D (left) strands will be part of the same component.

Proof. Note that adding two half twists to a tower does not change which components the strands belong to, as the top strands will still connect to the same bottom strands as if the two half twists were not present. For a tower with 1 half twist, the the A and D strands will be part of the same component, and the B and C strands will be part of the same component. Then for an odd tower, the same holds. Similarly, for a tower with 2 half twists, the A and C strands will be part of the same component, and the B and D strands will be part of the same component, so the same holds for an even tower. □

Corollary 1. If the strands on either side of an odd tower belong to the same component, then the entire odd tower belongs to the same component.

Given this definition for the strands of a pretzel tower, we will call the arc joining A of q_1 and B of q_n in the pretzel link $P(q_1, \cdots, q_n)$ the *upper closing arc.* The arc joining C of q_1 and D of q_n is similarly the *lower closing arc.*

We are interested in classifying which pretzel links are the result of strong fusion; thus, we are only interested in pretzel links with at least two components. A pretzel link $P(q_1, \cdots, q_n)$ has more than one component exactly when one of the following holds: (a) at least two of $q_1, \cdots, q_n$ are even (b) none of $q_1, \cdots, q_n$ are even and n is even [7].

3.1. CASE: Three or more even towers.

Theorem 3.1. *Given three or more non-zero even towers, a pretzel link is not the result of strong fusion.*

Proof. Take any even tower with q_j crossings within the pretzel link. Begin identifying components of the link by tracing the strand attached to B of the even tower and coloring it blue. Since the tower is even, D is also colored blue. Then both left strands of the next tower are colored blue. Now follow the subsequent towers:

- If the next tower is odd, since the left strands of an odd tower belong to the same component, then the right strands belong to the same component by the lemma, and the entire odd tower is part of the same blue component. Thus the presence of odd towers does not affect the components.
- If the next tower is even with q_k crossings, then the right strands are part of the same component, and the component is closed.

Now the blue component, denoted L_i, links with the rest of the pretzel link at the two even towers that it is involved in. However, since there are at least three even towers, the components other than the L_i component at those towers are distinct from each other. Denote those components L_j and L_k where they link with L_i at the towers with crossings q_j and q_k respectively. Then the linking number of L_i with either of those two components will be equal to either $|q_k/2|$ or $|q_j|/2$, where sign is determined by orientation. But for a component to be the distinguished component of strong fusion, it must have trivial linking number with all other components and thus must meet other components in trivial towers with zero half twists. □

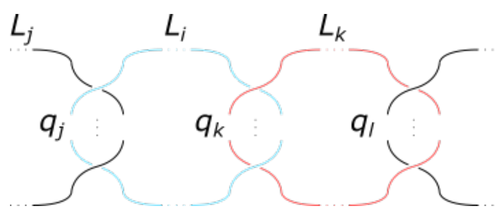


FIGURE 11. Pretzel link with at least 3 even towers. The even towers are denoted q_j, q_k, q_l, and the components of the link are denoted L_j, L_i, L_k

3.2. **CASE: Exactly Two Even Towers.** If a pretzel link has exactly two even towers, then it has exactly two components.

Lemma 4. If a pretzel link has exactly two even towers, then it has exactly two components.

Proof. Let $P(q_1, \cdots, q_m, q_{m+1}, \cdots, q_n)$ be a pretzel link with only q_1, q_m even. Consider the right strand of q_1. Since the BD strand of q_1 belongs to the same component (red), the AC strands of q_2 will belong to the same component. Then both strands of q_2 will belong to the red component, and so q_2 belongs completely to the red component (Corollary 1). Similarly, q_k is red for $1 < k < m$. At q_m, the left strand belongs to the red component. Color the other (right) strand of q_m blue. Since the top and bottom of the right strand belong to the same component, the q_{m+1} tower, which is odd, will completely belong to the same component (Corollary 1). Thus, q_k is blue for $m < k \leq n$. The q_n tower will be completely colored blue, so the left strand of q_1 will be blue. Therefore, there are exactly two components in a pretzel link with exactly two even towers. □

We are interested in when one of those components is the unknot.

Proposition 4. *A pretzel link $P(q_1, q_2, ..., q_n)$ with exactly two even towers and some number of odd towers is the result of strong fusion only if it can be written in the form:*

- $P(e, 2j_1 + 1, \cdots, 2j_s + 1, e, \pm 1, \cdots, \pm 1)$ *where* $e = \pm 2$ *given that the total number of towers is odd,*
- $P(e, 2j_1 + 1, \cdots, 2j_s + 1, -e, \pm 1, \cdots, \pm 1)$ *where* $e = \pm 2$ *given that the total number of towers is even.*

Proof. Consider $P(q_1, ..., q_n)$ with exactly two even towers. We may assume the even towers are q_1 and q_m where $1 \leq m \leq n$. Then $P(q_1, ..., q_n)$ has two components:

$$P_1 = \mathop{\#}_{k=2}^{m-1} T(k, 2)$$

and

$$P_2 = \mathop{\#}_{k=m+1}^{n} T(k, 2)$$

where $T(k, 2)$ denotes the torus knot. Note $T(k, 2)$ is a non-trivial knot for any odd $k \neq \pm 1$ (Kawauchi Theorem 2.2.2 [6]). Thus, in order for one component to be the distinguished component of strong fusion, the towers belonging to at least one component can only have values ± 1. Without loss of generality, we may assume $q_k = \pm 1$ for all $m < k \leq n$ so that the pretzel link is of the form

$$P(q_1, ..., q_n) = P(q_1, ..., q_m, \pm 1, ..., \pm 1).$$

Then P_2 is unknotted.

We then have that the two components of $P(q_1, ..., q_n)$ only cross at the towers q_1 and q_m, and so the linking number is entirely dependent on the values of q_1 and q_m. Consider the number of (odd) towers between q_1 and q_m, which we call $s = m - 2$, and the number of (odd) towers between q_m and q_n, which we call $t = n - (m + 1)$.

Now consider the right strand BD passing through the tower q_1 and note that it may be oriented in some vertical manner. In the following odd towers, the right strands will be oriented in the opposite direction from the previous tower's right strand. Hence, if s is odd then the BD strand will pass through the second even tower q_m through the AC strand in the same vertical direction as it passes through the q_1 tower. Meanwhile, if s is even, then the strand will pass through the q_m tower the opposite direction. Similarly the parity of t determines if the AC strand passing through tower q_1 passes through the tower q_m through the BD strand with the same (if t odd) or opposite (if t even) vertical direction. See Figure 12.

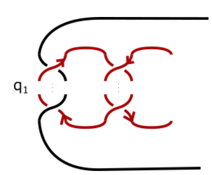


FIGURE 12. Effect of an odd tower on the orientation

Thus if n is odd, then s and t have opposite parity and hence if the q_1 tower contributes $\pm\frac{q_a}{2}$ (depending on orientation) to linking number, then the second even tower q_m will contribute $\mp\frac{q_b}{2}$ respectively to linking number. So $lk(L_1, L_2) = \pm\frac{q_a - q_b}{2}$. Meanwhile, if n is even, then s and t will have the same parity and thus $lk(L_1, L_2) = \pm\frac{q_a + q_b}{2}$.

By Proposition 2, in order for $P(q_1, ..., q_n)$ to be the result of strong fusion, $lk(L_1, L_2)$ must be zero, and thus $q_a = q_b$ if n is odd and $q_a = -q_b$ if n is even.

Finally, we may perform a dichromatic resolution on a pretzel links with exactly two even towers and some number of odd towers. Consider its distinguished component (P_2) to be green and the other component (P_1) to be red. As argued earlier in the proof, both strands of an odd tower will belong to the same component by Corollary 1. Additionally, both strands in any odd tower will have the same vertical direction. Thus, we may resolve the red crossings of the odd tower, smoothing them all out.

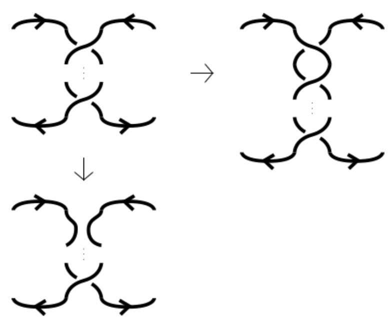


FIGURE 13. Resolution of a single crossing in an odd tower

This leaves two red components that have linking number $\pm|q_1|/2 = \pm|q_m|/2$ with the green component, or a component $a^{\pm q_1/2}a^{\pm q_m/2}$ in the final resolution. Thus, by Proposition 3 we must have $q_1 = q_m = \pm 2$. □

We use the properties of pretzel links introduced earlier to finding the strong fusion band in the proposed general form and thus prove the converse of the previous proposition.

Proposition 5. *A pretzel link $P(q_1, q_2, ..., q_n)$ of the following forms*

- $P(e, 2j_1 + 1, \cdots, 2j_s + 1, e, \pm 1, \cdots, \pm 1)$ *where $e = \pm 2$ given that the total number of towers is odd,*
- $P(e, 2j_1 + 1, \cdots, 2j_s + 1, -e, \pm 1, \cdots, \pm 1)$ *where $e = \pm 2$ given that the total number of towers is even.*

is the result of strong fusion.

Proof. Take the first case with $P(e, 2j_1+1, \cdots, 2j_s+1, e, \pm 1, \cdots, \pm 1)$ where $e = \pm 2$ given that the n is odd. By the cycling lemma 1,

$P(e, 2j_1 + 1, \cdots, 2j_s + 1, e, \pm 1, \cdots, \pm 1) = P(2j_1 + 1, \cdots, 2j_s + 1, e, \pm 1, \cdots, \pm 1, e)$

By Lemma 2, it is possible to move the ± 1 towers such that

$P(e, 2j_1+1, \cdots, 2j_s+1, -e, \pm 1, \cdots, \pm 1) = P(\pm 1, \cdots, \pm 1, 2n_1+1, \cdots, 2n_m+1, -e, e)$

Now this looks like

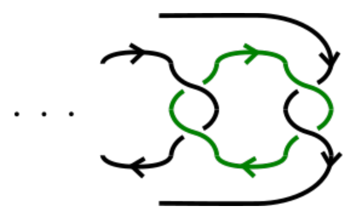


FIGURE 14.

For the n even case,

$P(2j_1 + 1, \cdots, 2j_s + 1, e, \pm 1, \cdots, \pm 1, e) = P(\pm 1, \cdots, \pm 1, 2j_1 + 1, \cdots, 2j_s + 1, e, e)$

So we have

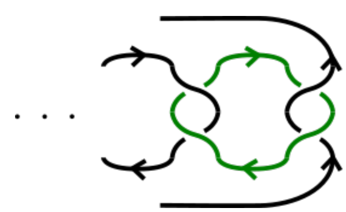


FIGURE 15.

□

Combining propositions 4 and 5, we obtain the following result.

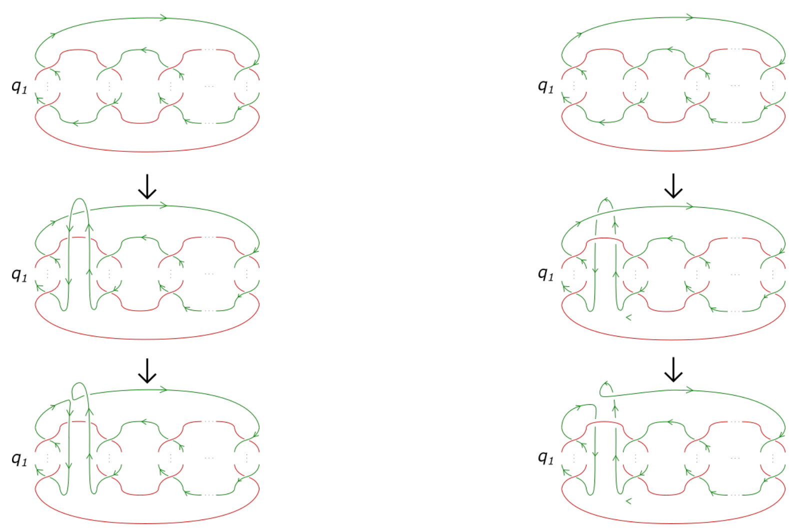
Theorem 3.2. *A pretzel link $P(q_1, q_2, ..., q_n)$ with exactly two even towers and some number of odd towers is the result of strong fusion if and only if it can be written in the form:*

- $P(e, 2j_1 + 1, \cdots, 2j_s + 1, e, \pm 1, \cdots, \pm 1)$ *where* $e = \pm 2$ *given that the total number of towers is odd,*
- $P(e, 2j_1 + 1, \cdots, 2j_s + 1, -e, \pm 1, \cdots, \pm 1)$ *where* $e = \pm 2$ *given that the total number of towers is even.*

3.3. CASE: Even number of odd towers (no even towers).

Proposition 6. *A pretzel link $P(q_1, ..., q_n)$ with n even and q_k odd for $k \in 1, \cdots, n$ has a dichromatic resolution tree that includes paths in which one resolution is a band sum with $a^{(q_k+1)/2}$ and another path which includes $a^{(q_k-1)/2}$ in the resolution for any $k \in 1, \cdots, n$.*

Proof. Without loss of generality it is sufficient to prove this in the case $k = 1$ as we can shift the towers (Lemma 1). Take the strand D attached at the bottom right of q_1. Perform a dichromatic resolution by moving upwards the arc connecting D of q_1 with C of q_2. Then there exist two resolution paths: one where the arc is pulled above the upper closing arc of the pretzel link (Figure 16a) and one where the arc is pulled below the upper closing arc (Figure 16b). The resolutions will have components $a^{(q_1+1)/2}$ and $a^{(q_1-1)/2}$ respectively. □



(A) Above the upper closing arc (B) Below the upper closing arc

FIGURE 16. Resolution paths of a pretzel link in Case 3

Corollary 2. A pretzel link $P(q_1, ..., q_n)$ with no even towers and an even number of odd towers is the result of strong fusion only when $q_k = \pm 1$ for all $k \in 1, \cdots, n$.

Proof. If a link is the result of strong fusion, then the only nontrivial components it may have are $a^1 \# a^{-1}$ [5]. Thus the tower with q_k crossings in the pretzel link must have $q_k = \pm 1$. □

Proposition 7. *A pretzel link $P(q_1, ..., q_n)$, n even and q_i odd for all $i \in 1, \cdots, n$, with 2 components P_1 and P_2 has $lk(P_1, P_2) = \pm\frac{1}{2}\sum_{k=1}^{n} q_k$.*

Proof. Since all the towers are odd, the (green) component that contains the AC strand of q_1 will contain the BD strands of the q_j towers for j even and the AC strands of the q_k towers for k odd. Then, since n is even, the BD strand of q_n will belong to the green component, and thus the component is closed. Similarly, the (red) component that contains the BD strand of q_1 will contain the AC strands of the q_j towers, j even, and the BD strands of the q_k towers for k odd. Then, since n is even, the AC strand of q_n will belong to the red component, and thus the component is closed. Then each tower will only involve twists between the two components and not one component by itself since the total number of odd towers are even and there are no even towers. The number of crossings with sign in each tower is simply the number of twists in the tower. So by the definition of linking number,

$$lk(P_1, P_2) = \pm\frac{1}{2}\sum_{k=1}^{n} q_k.$$

□

Theorem 3.3. *The only pretzel link with no even towers and an even number of odd towers that is the result of strong fusion is the trivial pretzel link.*

Proof. Suppose a pretzel link $P(q_1, ..., q_n)$ satisfies $q_k = \pm 1$ and $lk(P_1, P_2) = \pm\frac{1}{2}\sum_{k=1}^{n} q_k = 0$. Let the number of $+1$ towers be equal to x and the number of -1 towers be equal to y. Then $lk(P_1, P_2) = \pm\frac{x-y}{2}$. Additionally, $lk(P_1, P_2) = 0$; hence, $x = y$ and it is possible to pair every $+1$ tower with a -1 tower. Since all the towers have ± 1 crossings, there are two towers adjacent to each other that have opposite sign somewhere in the link. Then the towers cancel each other. But then the link is trivial, since all the towers are ± 1 and there are as many $+1$ towers as there are -1 towers. □

3.4. **Main Result.** Combining Theorems 3.1, 3.2, and 3.3, we arrive at the final theorem for the classification of all pretzel links that are the result of strong fusion.

Main Theorem. *A pretzel link $P(q_1, q_2, ..., q_n)$ is the result of strong fusion if and only if it is one of the following forms (up to permutation):*

1. *a trivial link with at least two components,*
2. *m is odd, the only even values are $q_j = q_k = \pm 2$, and each $p_i = \pm 1$ for all $j < i < k$,*
3. *m is even, the only even values are $q_j = -q_k = \pm 2$, and each $p_i = \pm 1$ for all $j < i < k$.*

4. Conclusion

The strong fusion operation, a variation of the fusion operation, on mathematical links has characteristic properties that have been previously derived. These properties give rise to three tests which allow us to identify when a link is not the result of strong fusion. A link that is the result of strong fusion must have an unknotted component that has linking number zero with every other component, and the dichromatic resolution of the link will result in elementary links of the form a^0 and $a\#a^{-1}$. While fusion has been well-studied in the past, strong fusion has been less studied, and this paper presents an original effort to classify exactly when a pretzel link is the result of strong fusion. By applying the three tests in conjunction with the known properties of pretzel links, we derive the main theorem, which connects pretzel links, a well-studied family of links, with the lesser-known operation of strong fusion of links.

References

[1] Colin Conrad Adams. *The knot book. An elementary introduction to the mathematical theory of knots*. New York: Freeman, 1994. XIII, 306. ISBN: 071672393X.

[2] D. Buck and K. Ishihara. "Coherent Band Pathways Between Knots and Links". In: *Journal of Knot Theory and Its Ramifications* 24 (2015).

[3] Jim Hoste and Mark E. Kidwell. "Dichromatic Link Invariants". In: *Transactions of the American Mathematical Society* 321.1 (1990), pp. 197–229. ISSN: 0002-9947. DOI: 10.1090/s0002-9947-1990-0961623-2.

[4] Uwe Kaiser. "Geometric Properties of Strong Fusion". In: *J. Knot Theory Ramif* 01 (1992), pp. 297–302. ISSN: 0218-2165. DOI: 10.1142/s0218216592000173.

[5] Uwe Kaiser. "Strong Band Sum and Dichromatic Invariants". In: 74.1 (1992), pp. 237–251. ISSN: 0025-2611. DOI: 10.1007/bf02567670.

[6] Akio Kawauchi. *A Survey of Knot Theory*. Basel, 1996. URL: http://dx.doi.org/10.1007/978-3-0348-9227-8.

[7] Candice Price. "Coloring Invariant and Determinants". In: (2003). URL: http://educ.jmu.edu/~taalmala/OJUPKT/candiceprice.pdf.