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A Curve Satisfying t/K=s with Constant K>0

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A Curve Satisfying $\tau/\kappa = s$ with constant $\kappa > 0$

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ABSTRACT
In the present paper, we investigate a space curve in which the curvature is constant and the torsion is a linear function. The aim of this paper is to find an explicit formula for this space curve when the ratio of the torsion to the curvature is a linear function when the curvature is constant.

KEYWORDS
Space Curve, Curvature, Torsion, General Helix, Frenet Frame, Series Solution, Rectifying Curve

1. INTRODUCTION

The structure of a helix is a significant, often studied field in differential geometry. A curve of a constant slope or general helix is defined by the property that its tangent vector field makes as a constant angle with a fixed straight line (the axis of the general helix) in Euclidean space $\mathbb{R}^3$. A prevalent result stated by M. A. Lancret[1] in 1802 and first proved by B. de Saint Venant in 1845 is that a necessary and sufficient condition is that a curve be a general helix in which the ratio of curvature to torsion is constant. Mathematicians know that a general helix has a constant ratio of torsion to curvature, but this ratio can be further studied by considering different relationship between the curvature and the torsion, such as what happens when the ratio of torsion to curvature is a linear function. In fact, B. Y. Chen[2] defined the conditions for such a case. The purpose of this paper is to investigate an explicit formula for space curves when the ratio of torsion to curvature is a linear function when the curvature is constant. To do this, we use a series of solutions to the third order differential equation that has polynomial coefficients.

2. PRELIMINARIES[3,4]

We consider a smooth curve $\alpha : I \to \mathbb{R}^3, \alpha = \alpha(t)$ such that $\alpha'(t) \neq 0$ and α' is continuous for any $t \in I$. The unit tangent vector field $\mathrm{T}(t) = \dfrac{\alpha'(t)}{|\alpha'(t)|}$ is well defined. After taking reparametrization with respect to arc length s, we may assume α is a unit speed curve i.e. the magnitude of derivative $\alpha'(s)$ is 1.

Definition

The **curvature** κ is defined as $\kappa(s) = |\mathrm{T}'(s)|$.

The curvature of a curve measures how fast a curve changes its direction at a specific point.

Alternatively, we have another formula for the curvature $\kappa(t) = \dfrac{|T'(t)|}{|\alpha'(t)|}$ by chain rule and the principle normal vector of a unit speed curve α is defined as $\mathrm{N}(t) = \dfrac{\mathrm{T}'(t)}{|\mathrm{T}'(t)|} = \dfrac{1}{\kappa(t)}\mathrm{T}'(t)$.

Note that the unit vectors T and N are perpendicular to each other. However, since the vectors are defined in $\mathbb{R}^3$, there is the third unit vector in $\mathbb{R}^3$, called the binormal vector which is orthogonal to both T and N : $\mathrm{B} = \mathrm{T} \times \mathrm{N}$.

Then these three vectors {T, N, B} form an orthonormal basis in $\mathbb{R}^3$, which is called the Frenet frame.

Let us investigate the derivatives of {T, N, B}: $\mathrm{B}' = (\mathrm{T} \times \mathrm{N})' = \mathrm{T}' \times \mathrm{N} + \mathrm{T} \times \mathrm{N}'$.

From the definition, we know that $\mathrm{T}' = \kappa\mathrm{N}$ and, $\mathrm{N} \times \mathrm{N} = 0$, and then we get $\mathrm{B}' = \mathrm{T} \times \mathrm{N}'$.

Thus we have the following definition.

Definition

Let $\alpha(t)$ be a unit speed curve. Then, the torsion of α is defined as the function $\tau(t)$ where $\mathrm{B}'(t) = -\tau(t)\mathrm{N}(t)$.

Note, by taking the dot product with $\mathrm{N}(t)$, we define τ by $\tau = -\langle \mathrm{B}', \mathrm{N} \rangle$.

Since we know that T, N, B are mutually perpendicular to each other, we write $\mathrm{N} = \mathrm{B} \times \mathrm{T}$. Using $\mathrm{B}' = -\tau\mathrm{N}$ and $\mathrm{T}' = \kappa\mathrm{N}$, we derive the following formula:

$$\mathrm{N}' = \mathrm{B}' \times \mathrm{T} + \mathrm{B} \times \mathrm{T}' = -\tau\mathrm{N} \times \mathrm{T} + \mathrm{B} \times \kappa\mathrm{N} = -\kappa\mathrm{T} + \tau\mathrm{B}$$

Frenet-Serret Theorem

The Frenet frame {T, N, B} satisfies the following derivatives given by,

$$\mathrm{T}' = \kappa\mathrm{N}$$
$$\mathrm{N}' = -\kappa\mathrm{T} + \tau\mathrm{B}$$
$$\mathrm{B}' = -\tau\mathrm{N}$$

3. MAIN RESULTS

Assume that $\kappa(s)=|T'(s)|>0$ is a constant and $\tau = \kappa s$ by our assumption.

It will be useful to reparametrize α by a parameter t given by:

$$t(s)=\int_0^s \kappa d\sigma = \kappa s.$$

From the Frenet-Serret formulas, we have the following,

$$\frac{dT}{dt}=\frac{dT}{ds}\cdot\frac{ds}{dt}=\kappa N\cdot\frac{1}{\kappa}=N,\quad \frac{dN}{dt}=-T+\frac{t}{\kappa}B,\quad \frac{dB}{dt}=-\frac{t}{\kappa}N$$ **Formula 1.**

By taking the second and the third derivative of N, we get

$$\frac{d^2N}{dt^2}=-\frac{dT}{dt}+\frac{B}{\kappa}+\frac{t}{\kappa}\frac{dB}{dt}=-N+\frac{B}{\kappa}+(-\frac{t^2}{\kappa^2}N)$$

$$\frac{d^3N}{dt^3}=-\frac{dN}{dt}+\frac{1}{\kappa}\frac{dB}{dt}-(\frac{2t}{\kappa^2}N)-\frac{t^2}{\kappa^2}\frac{dN}{dt}=-(1+\frac{t^2}{\kappa^2})\frac{dN}{dt}-\frac{3t}{\kappa^2}N$$

Then we have a third order differential equation on $N = N(t)$:

$$\frac{d^3N}{dt^3}+(1+\frac{t^2}{\kappa^2})\frac{dN}{dt}+\frac{3t}{\kappa^2}N=0$$

Since $t = 0$ is an ordinary point for this linear DE, there exists a series solution in a neighborhood of $t = 0$ so that we can substitute $N=\sum_{n=0}^{\infty}a_n t^n$ into the equation to determine the coeeficient vectors a_n's.

Now we have

$$\sum_{n=3}^{\infty}n(n-1)(n-2)a_n t^{n-3}+(1+\frac{t^2}{\kappa^2})\sum_{n=1}^{\infty}na_n t^{n-1}+\frac{3t}{\kappa^2}\sum_{n=0}^{\infty}a_n t^n=0$$

By taking index shift and expanding the equation, we get

$$\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}t^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+\frac{1}{\kappa^2}\sum_{n=2}^{\infty}(n-1)a_{n-1}t^n+\frac{3}{\kappa^2}\sum_{n=1}^{\infty}a_{n-1}t^n=0$$

It becomes

$$(a_1+6a_3)+(\frac{3a_0}{\kappa^2}+2a_2+24a_4)t+\sum_{n=2}^{\infty}[(n+3)(n+2)(n+1)a_{n+3}+(n+1)a_{n+1}+\frac{n+2}{\kappa^2}a_{n-1}]t^n=0$$

Then we have $a_1+6a_3=0$ and $\frac{3a_0}{\kappa^2}+2a_2+24a_4=0$.

That is, $a_3=-\frac{1}{6}a_1$ and $a_4=-\frac{1}{12}a_2-\frac{1}{8\kappa^2}a_0$ respectively.

Furthermore, for $n = 2$, $(5\cdot4\cdot3)a_5+3a_3+\frac{4}{\kappa^2}a_1=0$ which gives $a_5=(\frac{1}{120}-\frac{1}{15\kappa^2})a_1$.

When $n = 3$, $(6\cdot5\cdot4)a_6+4a_4+\frac{5}{\kappa^2}a_2=0$ which gives $a_6=(\frac{1}{360}-\frac{1}{24\kappa^2})a_2+\frac{a_0}{240\kappa^2}$.

Continuing this process, we are able to determine the coefficients for the series solution for

$$N(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots$$

$$= a_0 + a_1 t + a_2 t^2 + (-\frac{1}{6}a_1)t^3 + (-\frac{1}{12}a_2 - \frac{1}{8\kappa^2}a_0)t^4 + (\frac{1}{120} - \frac{1}{15\kappa^2})a_1 t^5 + [(\frac{1}{360} - \frac{1}{24\kappa^2})a_2 + \frac{a_0}{240\kappa^2}]t^6 + \cdots$$

$$= a_0[1 - \frac{1}{8\kappa^2}t^4 + \frac{1}{240\kappa^2}t^6 + (\frac{1}{384\kappa^4} - \frac{1}{13440\kappa^2})t^8 + \cdots] + a_1[t - \frac{1}{6}t^3 + (\frac{1}{120} - \frac{1}{15\kappa^2})t^5 +$$

$$(-\frac{1}{5040} + \frac{2}{315\kappa^2})t^7 + \cdots] + a_2[t^2 - \frac{1}{12}t^4 + (\frac{1}{360} - \frac{1}{24\kappa^2})t^6 + (\frac{5}{2016\kappa^2} - \frac{1}{20160})t^8 + \cdots]$$

To find T from here, we can take $\int$N to have

$$T(t) = a_0[t - \frac{1}{40\kappa^2}t^5 + \frac{1}{1680\kappa^2}t^7 + (\frac{1}{3456\kappa^4} - \frac{1}{120960\kappa^2})t^9 + \cdots] + a_1[\frac{t^2}{2} - \frac{1}{24}t^4 + (\frac{1}{720} - \frac{1}{90\kappa^2})t^6 +$$

$$(-\frac{1}{40320} + \frac{1}{1260\kappa^2})t^8 + \cdots] + a_2[\frac{t^3}{3} - \frac{1}{60}t^5 + (\frac{1}{2520} - \frac{1}{168\kappa^2})t^7 + (\frac{5}{18144\kappa^2} - \frac{1}{181440})t^9 + \cdots] + \mathrm{b}$$

For some constant vectors a_0, a_1, a_2 and *b*.
Since $\alpha'(t) =$ T, from here, we take $\int T$ again and then

$$\alpha(t) = a_0[\frac{t^2}{2} - \frac{1}{240\kappa^2}t^6 + \frac{1}{13440\kappa^2}t^8 + (\frac{1}{34560\kappa^4} - \frac{1}{1209600\kappa^2})t^{10} + \cdots] + a_1[\frac{t^3}{6} - \frac{1}{120}t^5 +$$

$$(\frac{1}{5040} - \frac{1}{630\kappa^2})t^7 + (-\frac{1}{362880} + \frac{1}{11340\kappa^2})t^9 + \cdots] + a_2[\frac{t^4}{12} - \frac{1}{360}t^6 + (\frac{1}{20160} - \frac{1}{1344\kappa^2})t^8 +$$

$$(\frac{1}{36288\kappa^2} - \frac{1}{1814400})t^{10} + \cdots] + \mathrm{bt} + \mathrm{c}$$

For some constant vectors a_0, a_1, a_2, and *c*.

4. ANALYSIS

Because T and N are orthogonal to each other, it satisfies $\langle T, N\rangle = 0$ for any t.

It is true for any t so that $\langle a_0, b\rangle = 0$ since $N(0) = a_0$ and $T(0) = b$.

Also, by using the fact that T is a unit vector $\langle T, T\rangle = |T|^2 = 1$, we can get $\langle b, b\rangle = 1 = \langle a_0, a_0\rangle$.

We want to find the unknown constant vectors a_0, a_1, a_2, b, c so that we can determine the curve, $\alpha(t)$ completely.

Without loss of generality, we may assume that $\alpha(0) = c = \langle 0, 0, 0\rangle$ and $T(0) = b = \langle 0, 1, 0\rangle$, that is, $c = 0$ and $b = j$

And since $N(0) = a_0$, which is a unit vector orthogonal to b, let $a_0 = i = \langle 1, 0, 0\rangle$.

Also, we have $N'(0) = a_1$. By **Formula 1**, we have $N'(0) = -T(0) = -b = -j$.

Now, we have $a_0 = i$, $a_1 = -j$, $b = j$.

We take $t = 0$ into the second derivative and then we derive $N''(0) = 2a_2$.

By definition, $B(0) = T(0) \times N(0) = j \times i = -k$ and from **Formula 1**, we have an expression

$$N''(0) = -N(0) + \frac{1}{\kappa}B(0) = -i - \frac{k}{\kappa}.$$

Using two equations of N''(0), we can conclude that $a_2 = \langle -\frac{1}{2}, 0, -\frac{1}{\kappa}\rangle$.

Now, we have the following coefficient vectors

$$a_0 = \langle 1,0,0\rangle,\ a_1 = \langle 0,-1,0\rangle,\ a_2 = \langle -\frac{1}{2}, 0, -\frac{1}{\kappa}\rangle,\ b = \langle 0,1,0\rangle,\ c = \langle 0,0,0\rangle$$

Therefore, here is the summary of the result.

Theorem

A unit speed curve $\alpha(s)$ with torsion $\tau = \kappa s$ and a constant curvature $\kappa > 0$ has the expression, after taking some reparametrization,

$$\alpha(t) = \langle f_1(t) - \frac{1}{2} f_3(t), -f_2(t) + 1, -\frac{1}{\kappa} f_3(t)\rangle$$

$$\text{where } f_1(t) = \frac{t^2}{2} - \frac{1}{240\kappa^2}t^6 + \frac{1}{13440\kappa^2}t^8 + (\frac{1}{34560\kappa^4} - \frac{1}{1209600\kappa^2})t^{10} + \cdots,$$

$$f_2(t) = \frac{t^3}{6} - \frac{1}{120}t^5 + (\frac{1}{5040} - \frac{1}{630\kappa^2})t^7 + (-\frac{1}{362880} + \frac{1}{11340\kappa^2})t^9 + \cdots,$$

$$\& f_3(t) = \frac{t^4}{12} - \frac{1}{360}t^6 + (\frac{1}{20160} - \frac{1}{1344\kappa^2})t^8 + (\frac{1}{36288\kappa^2} - \frac{1}{1814400})t^{10} + \cdots.$$

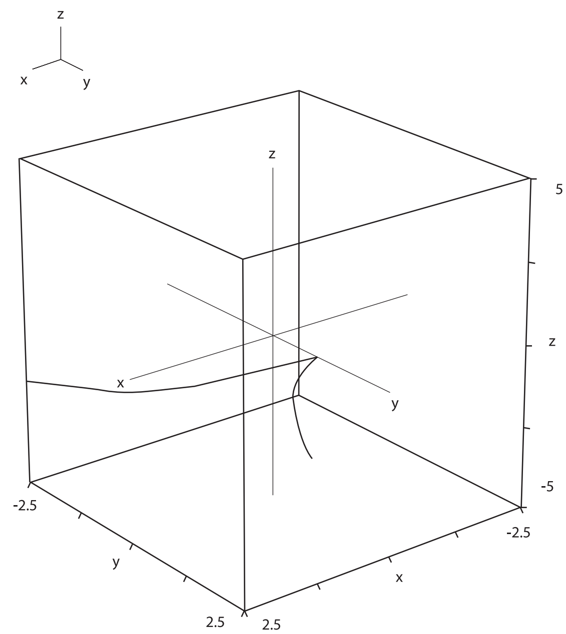

Figure 1. A graph of the curve.

Remark

(a) A space curve is defined as rectifying[2] if for some fixed point p, $\alpha(s) - \mathrm{p}$ is always contained in the plane spanned by $T(s)$ and $B(s)$. In the same paper, the author proved that a curve is rectifying if and only if $\frac{\tau}{\kappa} = as + b$ for constants a and b with $a \neq 0$.

(b) It was shown[2] that $\alpha(s)$ is rectifying if and only if it can be reparametrized to have the form $\alpha(t) = a\sec(t)\beta(t)$, where $\beta(t)$ is a unit speed curve on the unit sphere S^2 and $a > 0$.

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Ye Lim Seo is an undergraduate student at Andrews University. She has shown strong interest in mathematics and has received several math awards.

PRESS SUMMARY

The first chapter of undergraduate differential geometry is about the curves in 3-dimensional space. Because one of famous results about the general helix is that it has the constant ratio of torsion to curvature, a natural question is to think about the next simple case: the ratio of torsion to curvature as a linear function. Using the idea of series solution to the differential equation, one can find the curve explicitly with the extra condition that the curvature is a constant.