The Geometry of Curves

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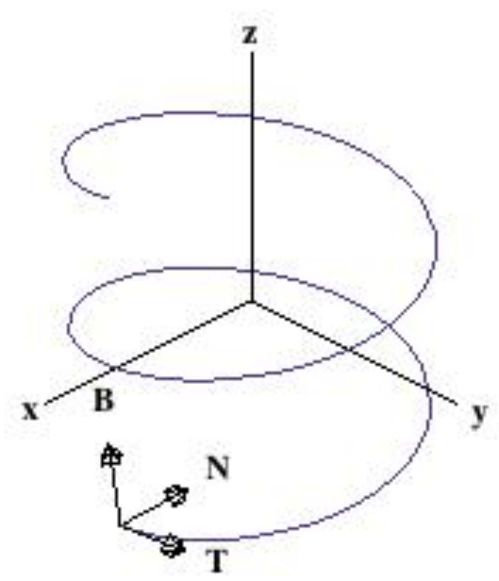
Introduction



The structure of helix is a significant field in the differential geometry studies, and it is profoundly studied and still studied over the period. A curve of constant slope or general helix is defined by the property that its tangent vector field makes a constant angle with a fixed straight line (the axis of the general helix) in Euclidean space .

A prevalent result stated by *M.A. Lancret* in 1802 and first proved by *B. de Saint Venant* in 1845 is: *A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant.*

Now we know that a general helix has a constant ratio of curvature to torsion, it can be further studied by considering different relationship between curvature and torsion. The purpose of this study is to investigate the behavior of the helix when the ratio of curvature to torsion is a linear function.

Frenet-Serret Theorem

The **curvature** is defined as $\kappa(s) = |T'(s)|$

The **unit tangent vector** of is denoted as $\mathrm{T}(t) = \dfrac{\alpha'(t)}{|\alpha'(t)|}$

The principle **normal vector** of is denoted as $\mathrm{N}(t) = \dfrac{\mathrm{T}'(t)}{|\mathrm{T}'(t)|} = \dfrac{1}{\kappa'(t)} \cdot \mathrm{T}'(t)$

Note that the unit vectors T and N are perpendicular to each other. However, since the vectors are defined in $\mathbb{R}^3$, we can easily assume that there is a third unit vector. This can be defined by the binomial which is orthogonal to both T and N .

$$\mathrm{B} = \mathrm{T} \times \mathrm{N}$$

Then, these three vectors, T , N , B , form orthonormal basis in R^3, which is called the Frenet Frame.

Definition:

Let $\alpha(t)$ be a unit speed curve. Then, the torsion of $\alpha(t)$ is denoted as the function $\tau(t)$ where $\mathrm{B}'(t) = -\tau(t)\mathrm{N}(t)$.

Note, by taking the dot product with $\mathrm{N}(t)$, using $\mathrm{N} \cdot \mathrm{N} = 1$, we define τ by $\tau = -\mathrm{B}' \cdot \mathrm{N}$.

Since we know that T, N, B are mutually perpendicular to each other, we write $\mathrm{N} = \mathrm{B} \times \mathrm{T}$. Then, we calculate N' by differentiating the previous relation.

$$\mathrm{N}' = \mathrm{B}' \times \mathrm{T} + \mathrm{B} \times \mathrm{T}'$$

Using $\mathrm{B}' = -\tau\mathrm{N}$ and $\mathrm{T}' = \kappa\mathrm{N}$, we derive following formula:

$$\mathrm{N}' = \mathrm{B}' \times \mathrm{T} + \mathrm{B} \times \mathrm{T}' = -\tau\mathrm{N} \times \mathrm{T} + \mathrm{B} \times \kappa\mathrm{N} = -\kappa\mathrm{T} \times \tau\mathrm{B}$$

The Frenet Frame {T, N, B} satisfies the following derivatives given by the definition:

$$T'(t) = \kappa(t)N(t)$$
$$N' = -\kappa(t)T(t) + \tau(t)B(t)$$
$$B'(t) = -\tau(t)N(t)$$

Research Question

We consider the case of which curvature is constant and torsion is a linear function. Investigate how the curve alters under these conditions.

Curvature and Torsion of Plane Curves

Assume that $\kappa(s) = |T'(s)|$ (constant) and $\tau = \kappa s$

From the Frenet Frame formulas, we have the following

$$\frac{d\mathrm{T}}{dt} = \frac{d\mathrm{T}}{ds} \cdot \frac{ds}{dt} = \kappa\mathrm{N} \cdot \frac{1}{\kappa} = \mathrm{N} \quad , \quad \frac{d\mathrm{N}}{dt} = -\mathrm{T} + t\mathrm{B} \quad , \quad \frac{d\mathrm{B}}{dt} = -t\mathrm{N}$$

Take the third derivative of N .

$$\frac{d^3\mathrm{N}}{dt^3} + (1+t^2)\frac{d\mathrm{N}}{dt} + 3t\mathrm{N} = 0$$

We take $\mathrm{N} = \sum_{n=0}^{\infty} a_n t^n$ to solve the third order differential equation.

Then we have $\sum_{n=3}^{\infty} n(n-1)(n-2)a_n t^{n-3} + (1+t^2)\sum_{n=1}^{\infty} n a_n t^{n-1} + 3t\sum_{n=0}^{\infty} a_n t^n = 0$.

By index shift and rearranging the above equation, we have

$$[(a_1 + 6a_3) + (3a_0 + 2a_2 + 24a_4)t] + \sum_{n=2}^{\infty}[(n+3)(n+2)(n+1)a_{n+3} + (n+1)a_{n+1} + (n+2)a_{n-1}]t^n = 0 .$$

To find the constant vectors that satisfy the equation, we know that $a_3 = -\dfrac{1}{6}a_1$ and $a_4 = -\dfrac{3a_0 + 2a_2}{24}$.

Also, we have to find the coefficients for each degree of the polynomial, and set them equal to zero.

Consider $n = 2$, then $(5 \cdot 4 \cdot 3)a_5 + 3a_3 + 4a_1 = 0$ which gives $a_5 = -\dfrac{7}{120}a_1$.

If $n = 3$, then $(6 \cdot 5 \cdot 4)a_6 + 4a_4 + 5a_2 = 0$ which gives $a_6 = \dfrac{1}{240}a_0 - \dfrac{7}{180}a_2$.

Continuing this process, we are able to substitute each a_n value into the polynomial equation, $a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots$.

Then, by rearranging the equation, we get the following.

$$\mathrm{N} = a_0\left(1 - \frac{1}{8}t^4 + \frac{1}{240}t^6 + \frac{17}{6720}t^8 + \cdots\right) + a_1\left(t - \frac{1}{6}t^3 - \frac{7}{120}t^5 + \frac{31}{5040}t^7 + \cdots\right) + a_2\left(t^2 - \frac{1}{12}t^4 - \frac{7}{180}t^6 + \frac{7}{2880}t^8 + \cdots\right)$$

To find T and $\alpha(t)$ from N, we take antiderivative of the above equation twice which gives the following equations.

$$\mathrm{T} = a_0\left(t - \frac{1}{40}t^5 + \frac{1}{1680}t^7 + \frac{17}{60480}t^9 + \cdots\right) + a_1\left(\frac{1}{2}t^2 - \frac{1}{24}t^4 - \frac{7}{720}t^6 + \frac{31}{40320}t^8 + \cdots\right) + a_2\left(\frac{1}{3}t^3 - \frac{1}{60}t^5 - \frac{1}{180}t^7 + \frac{7}{25920}t^9 + \cdots\right) + b$$

$$\alpha(t) = a_0\left(\frac{1}{2}t^2 - \frac{1}{240}t^6 + \frac{1}{13440}t^8 + \frac{17}{604800}t^{10} + \cdots\right) + a_1\left(\frac{1}{6}t^3 - \frac{1}{120}t^5 - \frac{1}{720}t^7 + \frac{31}{362880}t^9 + \cdots\right) + a_2\left(\frac{1}{12}t^4 - \frac{1}{360}t^6 - \frac{1}{1440}t^8 + \frac{7}{259200}t^{10} + \cdots\right) + bt + c$$

Analysis

Because T and N are orthonormal to each other, it satisfies $\mathrm{T} \cdot \mathrm{N} = 0$ for $\forall t$.

To simplify the equation, we take $t = 0$ for the equations T and N.

Then we have $\mathrm{N} = a_0$ and $\mathrm{T} = b$, where a_0 and b are constant vectors. Hence, we conclude that dot product of the those two constant vectors are zero since they are perpendicular to each other. i.e., $\langle a_0, b \rangle = 0$

Also, by using the fact that T is a unit tangent vector $\mathrm{T} \cdot \mathrm{T} = |\mathrm{T}|^2 = 1$, we conclude that the dot product of T and itself is equal to 1.

i.e., $\langle b, b \rangle = 1$

References

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