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### The Geometry of Curves

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polynomial, and set them equal to zero. denoted as the function  $\tau(t)$  where  $B'(t) = -\tau(t)N(t)$ .

Note, by taking the dot product with  $N(t)$ , using  $N \cdot N = 1$ , we define  $\tau$  by  $\tau = -B \cdot N$ .

# THE GEOMETRY OF CURVES **Ye Lim Seo**

## Frenet-Serret Theorem

The **curvature** is defined as  $\kappa(s) = |T'(s)|$ 

- *1. Barros, Manuel. "General Helices and A Theorem of Lancret." American Mathematical Society 125 (1997): 1503-1509. Print.*
- *2. Carmo, Differential geometry of curves and surfaces. Upper Saddle Rive, N.J,: Prentice-Hall, 1976. Print.*
- 3. Millman, Richard S., and George D. Parker. *Elements of differential geometry*. Englewood Cliffs, N.J.: Prentice-Hall, 1977. Print.

The **unit tangent vector** of is denoted as Τ(*t*) =  $\alpha'(t)$  $|\alpha'(t)|$ 

# **Introduction**

The structure of helix is a significant field in the differential geometry studies, and it is profoundly studied and still studied over the period. A curve of constant slope or general helix is defined by the property that its tangent vector field makes a constant angle with a fixed straight line (the axis of the general helix) in Euclidean space .



From the Frenet Frame formulas, we have the following  $N = N$ ,  $\frac{dV}{dt} = -T + tB$ , *d*Τ *dt* = *d*Τ *ds* ·<br>· *ds dt*  $= \kappa N \cdot$ 1 κ  $=$  N *d*Ν *dt*  $=-T + tB$   $\qquad \underline{dB}$ *dt*  $=-tN$ 

Take the third derivative of  $N$  .

References

The principle **normal vector** of is denoted as Ν(*t*) =  $T'(t)$  $|\mathbf{T}'(t)|$ = 1

To find the constant vectors that satisfy the equation, we know that  $a_3 = -\frac{1}{6}a_1$  and  $a_4 = -\frac{3a_0 + 2a_2}{24}$ . Also, we have to find the coefficients for each degree of the **Definition:**  Let  $\alpha(t)$  be a unit speed curve. Then, the torsion of  $\alpha(t)$  is Note that the unit vectors T and N are perpendicular to each other. However, since the vectors are defined in  $\mathbb{R}^3$ , we can easily assume that there is a third unit vector. This can be defined by the binomial which is orthogonal to both  $T$  and  $N$ .  $K'(t)$ ⋅Τ'(*t*)  $B = T \times N$ Then, these three vectors,  $T$ ,  $N$ ,  $B$ , form orthonormal basis in  $R^3$ , which is called the Frenet Frame. We take  $N = \sum a_n t^n$  to solve the third order differential equation. Then we have  $\sum n(n-1)(n-2)a_nt^{n-3} + (1+t^2)\sum na_nt^{n-1} + 3t\sum a_nt^n = 0$ . By index shift and rearranging the above equation, we have  $[(a_1 + 6a_3) + (3a_0 + 2a_2 + 24a_4)t] + \sum_{n=2}^{3} [(n+3)(n+2)(n+1)a_{n+3} + (n+1)a_{n+1} + (n+2)a_{n-1}]t^n = 0$ *dt n*=0 ∞ ∑  $n(n-1)(n-2)a_nt^{n-3}$ *n*=3 ∞  $\sum n(n-1)(n-2)a_nt^{n-3} + (1+t^2)\sum na_nt^{n-1}$ *n*=1 ∞  $\sum$ <sup>*n*</sup> $a_n t^{n-1} + 3t \sum a_n t^n$ *n*=0 ∞  $\sum a_n t^n = 0$ *n*=2 ∞  $\sum (n+3)(n+2)(n+1)a_{n+3} + (n+1)a_{n+1} + (n+2)a_{n-1}$ ] $t^n = 0$ 6  $a_1$  and  $a_4$  $=-\frac{3a_0+2a_2}{24}$ 24

The Frenet Frame {T, N, B} satisfies the following derivatives given by the definition:

$$
T'(t) = \kappa(t)N(t)
$$

$$
N' = -\kappa(t)T(t) + \tau(t)B(t)
$$

$$
B'(t) = -\tau(t)N(t)
$$

 A prevalent result stated by *M.A. Lancret* in 1802 and first proved by *B. de Saint Venant* in 1845 is: *A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant.* 

Since we know that T, N, B are mutually perpendicular to each other, we write  $N = B \times T$ . Then, we calculate N' by differentiating the previous relation.

 $N' = B' \times T + B \times T'$ 

Using  $B' = -\tau N$  and  $T' = \kappa N$ , we derive following formula:  $N' = B' \times T + B \times T' = -\tau N \times T + B \times \kappa N = -\kappa T \times \tau B$ 

### Research Question

Now we know that a general helix has a constant ratio of curvature to torsion, it can be further studied by considering different relationship between curvature and torsion. The purpose of this study is to investigate the behavior of the helix when the ratio of curvature to torsion is a linear function. We consider the case of which curvature is constant and torsion is a linear function. Investigate how the curve alters under these conditions.

# Curvature and Torsion of Plane Curves

Assume that  $\kappa (s) = |T'(s)|$  (constant) and  $\tau$  =  $\kappa s$ 

$$
\frac{d^3N}{dt^3} + (1+t^2)
$$



 $N = a_0 \left( 1 - \frac{1}{8} \right)$ 8  $t^4$  + 1 240  $t^6 +$ 17 6720  $\left(1-\frac{1}{2}t^4+\frac{1}{240}t^6+\frac{17}{6720}t^8+\cdots\right)$ ⎝  $\left(1-\frac{1}{8}t^4+\frac{1}{240}t^6+\frac{17}{6720}t^8+\cdots\right)$ ⎠  $\bigg] + a_1 \bigg( t - \frac{1}{6}$ 6  $t^3 - \frac{7}{12}$ 120  $t^5 +$ 31 5040  $\left(t-\frac{1}{5}t^3-\frac{7}{120}t^5+\frac{31}{5040}t^7+\cdots\right)$ ⎝  $\left(t-\frac{1}{6}t^3-\frac{7}{120}t^5+\frac{31}{5040}t^7+\cdots\right)$ ⎠  $\bigg] + a_2 \bigg( t^2 - \frac{1}{16}$ 12  $t^4 - \frac{7}{18}$ 180  $t^6 +$  $\int t^2 - \frac{1}{12} t^4 - \frac{7}{120} t^6 + \frac{7}{28000} t^8 + \cdots$ ⎝  $\left(t^2 - \frac{1}{12}t^4 - \frac{7}{180}t^6 + \frac{7}{2880}t^8 + \cdots \right)$ If  $n = 3$ , then  $(6.5.4)a_6 + 4a_4 + 5a_2 = 0$  which gives  $a_6 = \frac{1}{240}a_0 - \frac{7}{180}a_2$ . Continuing this process, we are able to substitute each *an* value into the polynomial equation,  $a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots$ . Then, by rearranging the equation, we get the following. 1 240  $a_0 - \frac{7}{10}$ 180  $a<sub>2</sub>$ To find  $T$  and  $\alpha(t)$  from N, we take antiderivative of the above equation twice which gives the following equations.  $T = a_0 \left( t - \frac{1}{40} \right)$ 40  $t^5 +$ 1 1680  $t^7 +$ 17 60480  $\left(t-\frac{1}{40}t^5+\frac{1}{1600}t^7+\frac{17}{60400}t^9+\cdots\right)$  $\left(t - \frac{1}{40}t^5 + \frac{1}{1680}t^7 + \frac{17}{60480}t^9 + \cdots\right) + a_1$ 1 2  $t^2 - \frac{1}{2}$ 24  $t^4 - \frac{7}{72}$ 720  $t^6 +$ 31 40320  $\left(\frac{1}{2}t^2 - \frac{1}{24}t^4 - \frac{7}{722}t^6 + \frac{31}{42222}t^8 + \cdots\right)$  $\left(\frac{1}{2}t^2 - \frac{1}{24}t^4 - \frac{7}{720}t^6 + \frac{31}{40320}t^8 + \cdots\right) + a_2$ 1 3  $t^3 - \frac{1}{6}$ 60  $t^5 - \frac{1}{18}$ 180  $t^7 +$  $\left(\frac{1}{2}t^3 - \frac{1}{60}t^5 - \frac{1}{100}t^7 + \frac{7}{25000}t^9 + \cdots\right)$  $\left(\frac{1}{3}t^3 - \frac{1}{60}t^5 - \frac{1}{180}t^7 + \frac{7}{25920}t^9 + \cdots\right) + b$  $\alpha(t) = a_0$ 1 2  $t^2 - \frac{1}{24}$ 240  $t^6 + \frac{1}{124}$ 13440  $t^8 + \frac{17}{6040}$ 604800  $\left(\frac{1}{2}t^2 - \frac{1}{240}t^6 + \frac{1}{12440}t^8 + \frac{17}{604000}t^{10} + \cdots\right)$  $\left(\frac{1}{2}t^2 - \frac{1}{240}t^6 + \frac{1}{13440}t^8 + \frac{17}{604800}t^{10} + \cdots\right) + a_1$ 1 6  $t^3 - \frac{1}{12}$ 120  $t^5 - \frac{1}{72}$ 720  $t^7 + \frac{31}{2620}$ 362880  $\left(\frac{1}{\epsilon}t^3 - \frac{1}{120}t^5 - \frac{1}{720}t^7 + \frac{31}{262000}t^9 + \cdots\right)$  $\left(\frac{1}{6}t^3 - \frac{1}{120}t^5 - \frac{1}{720}t^7 + \frac{31}{362880}t^9 + \cdots\right) + a_2$ 1 12  $t^4 - \frac{1}{26}$ 360  $t^6 - \frac{1}{14}$ 1440  $t^8 + \frac{7}{2500}$ 259200  $\left(\frac{1}{12}t^4 - \frac{1}{260}t^6 - \frac{1}{1440}t^8 + \frac{7}{250000}t^{10} + \cdots\right)$  $\left(\frac{1}{12}t^4 - \frac{1}{360}t^6 - \frac{1}{1440}t^8 + \frac{7}{259200}t^{10} + \cdots\right) + bt + c$ 

*d*Ν

 $+3tN = 0$ 







### **Analysis**

Because T and N are orthonormal to each other, it satisfies  $T \cdot N = 0$  for  $\forall t$ .

To simplify the equation, we take  $t = 0$  for the equations T and N. Then we have  $N = a_0$  and  $T = b$ , where  $a_0$  and  $b$  are constant vectors. Hence, we conclude that dot product of the those two constant vectors are zero since they are perpendicular to each other. i.e.,  $\langle a_0,b\rangle = 0$ 

Also, by using the fact that  $T$  is a unit tangent vector  $T \cdot T = |T|^2 = 1$ ,

we conclude that the dot product of T and itself is equal to 1.

i.e., 
$$
\langle b,b\rangle=1
$$